Monte Carlo methods

Exercise 1 the Gamma distribution

Let $k \in \mathbb{N}^*$. The Gamma probability distribution function with parameter k and $\lambda > 0$, denoted $\gamma(k, \lambda)$ is given by

$$f(x) = \frac{x^{k-1} e^{-x/\lambda}}{\lambda^k (k-1)!} \, 1_{\{x>0\}}.$$

The aim of this exercise is to estimate the value I_2 defined by

$$I_2 = \mathbb{E}\left[\frac{\ln(X)}{X}\right] \quad \textit{with} \quad X \sim \gamma(2, 1).$$

1. *Show that*
$$I_2 = \int_0^1 \ln\left(\ln\left(\frac{1}{u}\right)\right) du. \tag{1}$$
Is it possible with expression (1) to introduce a Hit & Miss Monte Carlo method in order to estimate I_2? Explain.

2. *Using (1), suggest a mean sampling for the estimation of I_2. Describe the convergence rate of this estimator.*

3. *Let $(X_n)_{n\geq 1}$ a sequence of independent random variables with distribution $\gamma(2, 1)$. We define Θ_N by*
$$\Theta_N = \frac{1}{N} \sum_{n=1}^{N} \frac{\ln(X_n)}{X_n}, \quad N \geq 1.$$
Is it judicious to use the estimator Θ_N in order to obtain an approximation of I_2 ? Explain.

4. *Let $\alpha > 0$. Show that $I_2 = \ln(\alpha) + \mathbb{E}[\ln Y]$ with $Y \sim \mathcal{E}(\alpha)$. Deduce a mean sampling for the estimation of I_2. Prove the existence of a polynomial function P_2 of degree 2 such that $\mathrm{Var}(Y) = P_2(\ln \alpha)$. What is the best (theoretical) choice for the value α? Explain.*