Approximation of exit times for one-dimensional linear diffusion processes.

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Abstract

In order to approximate the exit time of a one-dimensional diffusion process, we propose an algorithm based on a random walk. Such an algorithm was already introduced in both the Brownian context and the Ornstein-Uhlenbeck context, that is for particular time-homogeneous diffusion processes. Here the aim is therefore to generalize this efficient numerical approach in order to obtain an approximation of both the exit time and position for a general linear diffusion. The main challenge of such a generalization is to handle with time-inhomogeneous diffusions. The efficiency of the method is described with particular care through theoretical results and numerical examples.

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1 Introduction

In many domains, the simulation of the first exit time for a diffusion plays a crucial role. In reliability analysis, for instance, first passage times and exit times are directly related to lifetimes of engineering systems. In order to emphasize explicit expressions of the lifetime distribution, it is quite usual to deal with simplified models like Ornstein-Uhlenbeck processes. Indeed

they satisfy the mean reverting property which is essential for modeling degradation processes. In mathematical finance studying barrier options also requires to describe exit times since it is of prime interest to estimate if the underlying stock price stays in a given interval. In the simple Black-Scholes model, the distribution of the first exit time is well-known. In more complex models corresponding to general diffusion processes, such an explicit expression is not available and requires the use of numerical approximations.

Several methods have been introduced in order to approximate first exit times. The classical and most common approximation method is the Euler–Maruyama scheme based on a time-discretization procedure. The exit time of the diffusion process is in that case replaced by the exit time of the scheme. The approximation is quite precise but requires to restrict the study on a given fixed time interval on one hand and to describe precisely the probability for the diffusion to exit inbetween two consecutive nodes of the time grid on the other hand.

In this study, we aim to introduce a random walk in order to approximate the diffusion exit time from a given interval. Let us introduce $(X_t,\, t \geq 0)$ the unique solution of the stochastic differential equation:

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \quad t \geq 0,$$

where $(W_t,\, t \geq 0)$ stands for a one-dimensional Brownian motion. Let us also fix some interval $I = [a, b]$ which strictly contains the starting position $X_0 = x$. We denote by $\mathcal{T}$ the diffusion first exit time:

$$\mathcal{T} = \inf\{t \geq 0 :\ X_t \notin [a, b]\}.$$

Our approach consists in constructing a random walk $(T_n, X_n)_{n\geq 0}$ on $\mathbb{R}_+ \times \mathbb{R}$ which corresponds to a skeleton of the Brownian paths. In other words, the sequence (T_n, X_n) belongs to the graph of the trajectory. Moreover we construct the walk in such a way that (T_n, X_n) converges as time elapses towards the exit time and location $(\mathcal{T}, X_{\mathcal{T}})$. It suffices therefore to introduce a stopping procedure in the algorithm to achieve the approximation scheme. Of course, such an approach is interesting provided that (T_n, X_n) is easy to simulate numerically. For the particular Brownian case, the distribution of the exit time from an interval has a quite complicated expression which is difficult to use for simulation purposes (see, for instance [18]) whereas the exit distribution from particular time-dependent domains, for instance the spheroids also called *heat balls*, can be precisely determined. These time-

dependent domains are characterized by their boundaries:

$$\psi_{\pm}(t) = \pm\sqrt{t\log\left(\frac{d^2}{t}\right)}, \quad \text{for } t \in [0, d^2], \tag{1.1}$$

where the parameter $d > 0$ corresponds to the size of the spheroid. The first time the Brownian motion paths (t, W_t) exits from the domain $\{(t, x) : \ |x| \leq \psi_+(t)\}$, denoted by τ, is well-known. Its probability density function [10] is given by

$$p(t) = \frac{1}{d\sqrt{2\pi}}\sqrt{\frac{1}{t}\log\left(\frac{d^2}{t}\right)}, \quad t \geq 0. \tag{1.2}$$

It is therefore easy to generate such an exit time since τ and $d^2 U e^{-N^2}$ are identically distributed. Here U and N are independent random variables, U being uniformly distributed on $[0, 1]$ and N being a standard gaussian random variable. Let us notice that the boundaries of the spheroids satisfy the following bound:

$$|\psi_{\pm}(t)| \leqslant \frac{d}{\sqrt{e}}, \quad \forall t \in [0, d^2]. \tag{1.3}$$

This remark permits to explain the general idea of the algorithm. First we consider (T_0, X_0) the starting time and position of the Brownian paths, that is $(0, x)$. Then we choose the largest parameter d possible such that the spheroid starting in (T_0, X_0) is included in the domain $\mathbb{R}_+ \times [a, b]$. We observe the first exit time of this spheroid and its corresponding exit location, this couple is denoted by (T_1, X_1). Due to the translation invariance of the Brownian motion, we can construct an iterative procedure, just considering (T_1, X_1) like a starting time and position for the Brownian motion. So we consider a new spheroid included in the interval and (T_2, X_2) shall correspond to the exit of this second spheroid and so on. Step by step we construct a random walk on spheroids also called WOMS algorithm (Walk On Moving Spheres) which converges towards the exit time and position $(\mathcal{T}, W_{\mathcal{T}})$. This sequence is stopped as soon as the position X_n is close enough to the boundary of the considered interval. The idea of this algorithm lies in the definition of spherical processes and the walk on spheres introduced by Müller [12] and used in the sequel by Motoo [11] and Sabelfeld [16] [17]. It permits also in some more technical advanced way to simulate the first passage time for Bessel processes [4].

In this study, we focus our attention on diffusions which are strongly related to the Brownian motion: they can be expressed as functionals of the Brownian motion that is $X_t = f(t, W_t)$. The idea is to use this link to adapt the Brownian algorithm in an appropriate way. This link implies changes on the time-dependent domains for which the exit problem can be expressed in a simpler way. For these diffusion families, we present the random walk algorithm (WOMS), describe the approximation error depending on the stopping procedure and emphasize the efficiency of the method. We describe the mean number of generalized spheroids necessary to obtain the approximated exit time.

2 The algorithm

The Walk on Spheroids already introduced for the Ornstein-Uhlenbeck process in [7] permits to approximate the exit time in an efficient way. We aim to extend such numerical procedure to a wider class of stochastic processes. We focus our attention to the family of L-class diffusions (linear-type diffusions) which generalizes the Ornstein-Uhlenbeck processes. For such diffusions, all the coefficients are time-dependent. Moreover they are based on a strong relation with a one-dimensional Brownian motion.

2.1 L-class diffusions

This particular family of diffusions was already introduced in [19].

Definition 2.1 (L-class diffusions)**.** *We call L-class diffusion any solution of*

$$dX_t = (\alpha(t)X_t + \beta(t))dt + \tilde{\sigma}(t)dW_t \quad t \geq 0 \quad \textit{and} \quad X_0 = x_0, \tag{2.1}$$

where α, β and $\tilde{\sigma}$ are Hölder-continuous functions, $\tilde{\sigma}$ is furthermore positive and $(W_t)_{t\geq 0}$ is a one-dimensional Brownian motion.

Since α, β and $\tilde{\sigma}$ are measurable functions, the linear structure of the differential equation (2.1) implies both the existence and the uniqueness of a strong solution. Moreover the stochastic process does not explode a.s. if the initial data X_0 is square-integrable (see, for instance, Theorem 5.2.1 in [14]). Here the starting position is always deterministic, consequently the explosion phenomenon is never observed a.s.. It is possible to solve (2.1) in a classical way (see, for instance, Section 5.6 about linear equations in [9]).

Let us introduce

$$\theta(t) := -\int_0^t \alpha(s)ds. \tag{2.2}$$

An application of Itô's rule implies that the unique strong solution of (2.1) is given by

$$X_t = X_0\, e^{-\theta(t)} + e^{-\theta(t)} \int_0^t e^{\theta(s)}\beta(s)ds + e^{-\theta(t)} \int_0^t e^{\theta(s)}\tilde{\sigma}(s)dW_s, \quad t \geq 0.$$

This expression is actually not handy for the construction of the algorithm. Moreover, for simulation purposes, it suffices to deal with a stochastic process which has the same path distribution than the strong solution X, that means to consider weak solutions of (2.1). For these reasons, we would like, as for Ornstein-Uhlenbeck processes in [7], to transform the martingale part of the diffusion into a time-changed Brownian motion. Let θ the function defined in (2.2). Then the following process $(X_t,\ t \geq 0)$ based on the time-changed Brownian motion

$$X_t = f_L(t, x_0 + W_{\rho(t)}), \quad \forall t \geq 0, \tag{2.3}$$

is the unique weak solution of (2.1) with

$$f_L(t,x) = \frac{\tilde{\sigma}(t)}{\sqrt{\rho'(t)}}x + c(t), \quad c(t) = e^{-\theta(t)} \int_0^t \beta(s)e^{\theta(s)}ds$$
$$\text{and } \rho(t) = \int_0^t \tilde{\sigma}(s)^2 e^{2\theta(s)}ds. \tag{2.4}$$

This statement is essentially based on Itô's rule: it suffices to consider $f_L(t, x_0 + M_t)$ with the particular martingale $M_t := \int_0^t \sqrt{\rho'(s)}dW_s$.

Remark 2.2. *If the starting time associated to the study of the L-class diffusion is not the origin but another time t_0, then we also obtain an expression similar to* (2.3)*. Let Y_t be the unique weak solution of*

$$\begin{cases} dY_t = (\alpha(t+t_0)Y_t + \beta(t+t_0))dt + \tilde{\sigma}(t+t_0)dW_t, \quad t \geq 0 \\ Y_0 = X_{t_0}. \end{cases}$$

Then

$$Y_t = f_L(t+t_0, X_{t_0}e^{-\int_0^{t_0}\alpha(s)ds} + W_{\rho(t+t_0)-\rho(t_0)}) - e^{\int_{t_0}^{t+t_0}\alpha(s)ds}c(t_0). \tag{2.5}$$

2.2 Spheroids associated to a L-class diffusion process

Introducing the exit time of the spheroid.

We determine a specific spheroid for the diffusion by using the link with the time-changed Brownian motion. The boundaries of the spheroid associated to the diffusion starting at time t_0 in x_0 are denoted by $\psi^L_\pm(t;\, t_0, x_0)$ and the corresponding exit time is

$$\tau_L^{t_0} = \inf\{t > 0 :\ Y_t^L \notin [\psi^L_-(t;\, t_0, x_0), \psi^L_+(t;\, t_0, x_0)]\}.$$

Proposition 2.3. *Let us consider the spheroid starting in (t_0, X_{t_0}) with boundaries defined by*

$$\begin{aligned}\psi^L_\pm(t;\, t_0, X_{t_0}) = e^{-\theta(t+t_0)}\, \psi_\pm(\rho(t+t_0) - \rho(t_0)) + c(t+t_0)\\ + (X_{t_0} - c(t_0))e^{\int_{t_0}^{t+t_0} \alpha(s)ds}\end{aligned}$$

for all $t \geq 0$, then the associated exit time satisfies

$$\tau_L^{t_0} \stackrel{d}{=} \rho_L^{-1}(\tau + \rho_L(t_0)) - t_0 \tag{2.6}$$

where $\tau = \inf\{u > 0\ : W_u \notin [\psi_-(t), \psi_+(t)]\}$, $\psi_\pm$ being defined in (1.1).

Proof. By definition,

$$\begin{aligned}&\tau_L^{t_0} = \inf\{t > 0 : Y_t \notin [\psi^L_-(t;\, t_0, X_{t_0}), \psi^L_+(t;\, t_0, X_{t_0})]\}\\ &= \inf\Big\{t > 0 : e^{-\theta(t+t_0)}\, W_{\rho(t+t_0)-\rho(t_0)} + c(t+t_0) + (X_{t_0} - c(t_0))e^{\int_{t_0}^{t+t_0} \alpha(s)ds}\\ &\qquad\qquad \notin [\psi^L_-(t;\, t_0, X_{t_0}), \psi^L_+(t;\, t_0, X_{t_0})]\Big\}.\end{aligned}$$

Using $\psi^L_\pm$ introduced in the statement, we obtain the following expression for $\tau_L^{t_0}$:

$$\begin{aligned}&\inf\Big\{t > 0 : W_{\rho(t+t_0)-\rho(t_0)} \notin [\psi_-(\rho(t+t_0) - \rho(t_0)), \psi_+(\rho(t+t_0) - \rho(t_0))]\Big\}\\ &= \inf\{\rho^{-1}(u + \rho(t_0)) - t_0 > 0 :\ W_u \notin [\psi_-(u), \psi_+(u)]\}\\ &= \rho_L^{-1}(\tau + \rho_L(t_0)) - t_0,\end{aligned}$$

where $\tau = \inf\{u > 0 :\ W_u \notin [\psi_-(u), \psi_+(u)]\}$. □

Size determination of the spheroids

To define a WOMS algorithm for the L-class diffusions, we need to determine a suitable size for the spheroids in order to stay fully contained in the considered interval. Such size can be chosen by describing both the minimum and the maximum of the spheroid boundaries.
The size of the Brownian spheroid introduced in (1.1) depends on a scaling parameter $d > 0$, the support of the associated boundaries $\psi_\pm$ being therefore equal to $[0, d^2]$. Since the generalized spheroids used for L-class diffusion are directly linked to the Brownian ones, the parameter d also changes their size and the boundaries $\psi^L_\pm$ are defined on the support $[0, \rho^{-1}(d^2+\rho(t_0))-t_0]$. Let us now precise this parameter d.

Proposition 2.4. *Let $m > 0$ and $0 < \gamma < 1$. For any $(x_0, t_0) \in [a, b] \times \mathbb{R}^+$ we define a parameter $d = d(x_0, t_0)$ such that the spheroid associated to the L-class diffusion starting in (t_0, x_0) is totally included in $[a_{\gamma,x_0}, b_{\gamma,x_0}]$. Here a_{γ,x_0} and b_{γ,x_0} stands for $a_{\gamma,x} = a + \gamma(x - a)$ and $b_{\gamma,x} = b - \gamma(b - x)$. This parameter is given by*

$$d = \begin{cases} \frac{\min(1,\kappa_+)}{\Delta_m}(b_{\gamma,x_0} - x_0) \text{ if } b - x_0 \leqslant x_0 - a \\ \frac{\min(1,\kappa_-)}{\Delta_m}(x_0 - a_{\gamma,x_0}) \text{ if } x_0 - a \leqslant b - x_0 \end{cases} \tag{2.7}$$

where

$$\Delta_m = e^{-\theta(t_0)} e^{\int_{t_0}^{t_0+m} |\alpha(s)| ds} \left(\frac{1}{\sqrt{e}} + \sqrt{\int_{t_0}^{t_0+m} \frac{|\beta(s) + x_0\, \alpha(s)|^2}{\tilde{\sigma}(s)^2} ds} \right), \tag{2.8}$$

and $\kappa_\pm$ are defined by the following equations:

$$\kappa_+(b_{\gamma,x_0} - x_0) = \Delta_m \sqrt{\rho(t_0 + m) - \rho(t_0)}$$

and

$$\kappa_-(x_0 - a_{\gamma,x_0}) = \Delta_m \sqrt{\rho(t_0 + m) - \rho(t_0)}.$$

Remark 2.5.

- *The previous statement consists in finding d such that*

$$\begin{cases} d \leqslant \frac{1}{\Delta_m}(b_{\gamma,x_0} - x_0), \\ d \leqslant \frac{1}{\Delta_m}(x_0 - a_{\gamma,x_0}), \\ d^2 \leqslant \rho(t_0 + m) - \rho(t_0). \end{cases}$$

The last condition in particular leads to $t \leqslant m$ since ρ is a strictly increasing function.

- *It is possible to let m depend on the couple (t_0, x_0) which should permit to obtain bigger spheroids which are still included in the interval. Nevertheless for numerical purposes, such a procedure slows down drastically the algorithm we are going to present.*
- *The choice of the constant m is important, since it either slows down or speeds up the algorithm.*
- *It is also possible to replace x_0 by $\max(|a|, |b|)$ in the definition of Δ_m which therefore becomes independent of the starting position x_0. Nevertheless such a replacement slows down the algorithm.*

Proof of Proposition 2.4. Let us first point out an upper bound for ψ_+^L starting in (t_0, x_0). We first require that $d^2 \leqslant \rho(t_0 + m) - \rho(t_0)$. Let us define $\mathcal{R}_+^L(t) := \psi_+^L(t;\, t_0, x_0) - x_0$. By definition

$$\begin{aligned}\mathcal{R}_+^L(t) = e^{-\theta(t_0+t)} &\left(\psi_+(\rho(t+t_0) - \rho(t_0)) + \int_{t_0}^{t_0+t} \beta(s) e^{-\int_0^s \alpha(u)du} ds\right) \\ &+ x_0 \left(e^{\int_{t_0}^{t+t_0} \alpha(u)du} - 1\right).\end{aligned}$$

Recalling (1.3), we obtain

$$\mathcal{R}_+^L(t) \leq e^{-\theta(t_0+t)} \left(\frac{d}{\sqrt{e}} + \int_{t_0}^{t_0+t} \beta(s) e^{\theta(s)} ds\right) + x_0\, e^{-\theta(t_0+t)} \left(e^{\theta(t_0)} - e^{\theta(t+t_0)}\right)$$

$$\begin{aligned}\mathcal{R}_+^L(t) &\leqslant e^{-\theta(t_0+t)} \left(\frac{d}{\sqrt{e}} + \int_{t_0}^{t_0+t} \beta(s) e^{-\int_0^s \alpha(u)du} ds\right) \\ &\quad + x_0\, e^{-\theta(t_0+t)} \left(\int_{t_0}^{t+t_0} \alpha(s) e^{-\int_0^s \alpha(u)du}\right) \\ &\leqslant e^{-\theta(t_0)+\int_{t_0}^{t_0+t} |\alpha(s)|ds} \left(\frac{d}{\sqrt{e}} + \int_{t_0}^{t_0+t} \frac{|\beta(s) + x_0\, \alpha(s)|}{\tilde{\sigma}(s)} \tilde{\sigma}(s) e^{-\int_0^s \alpha(u)du} ds\right),\end{aligned}$$

since $\tilde{\sigma}$ is a positive function. Using Cauchy-Schwarz's inequality, we obtain the following upper-bound for $\mathcal{S}_+^L(t) := e^{\theta(t_0)} e^{-\int_{t_0}^{t_0+t} |\alpha(s)|ds}\, \mathcal{R}_+^L(t)$:

$$\begin{aligned}\mathcal{S}_+^L(t) &\leq \frac{d}{\sqrt{e}} + \left(\int_{t_0}^{t_0+t} \frac{|\beta(s) + x_0\, \alpha(s)|^2}{\tilde{\sigma}(s)^2} ds \int_{t_0}^{t_0+t} \tilde{\sigma}(s)^2 e^{-2\int_0^s \alpha(u)du} ds\right)^{1/2} \\ &= \frac{d}{\sqrt{e}} + \left(\int_{t_0}^{t_0+t} \frac{|\beta(s) + x_0\, \alpha(s)|^2}{\tilde{\sigma}(s)^2} ds\right)^{1/2} (\rho(t+t_0) - \rho(t_0))^{1/2}.\end{aligned}$$

Using $\rho(t_0+t)-\rho(t_0)\leqslant d^2$ and $t\leqslant m$, leads to

$$\begin{aligned}\mathcal{R}_+^L(t) &\leqslant de^{-\theta(t_0)+\int_{t_0}^{t_0+m}|\alpha(s)|ds}\left(\frac{1}{\sqrt{e}}+\sqrt{\int_{t_0}^{t_0+m}\frac{|\beta(s)+x_0\,\alpha(s)|^2}{\tilde{\sigma}(s)^2}ds}\right)\\ &= d\Delta_m.\end{aligned}$$

Under the condition $d\Delta_m+x_0\leqslant b_{\gamma,x_0}$, we observe that the spheroid belongs to the interval $d\Delta_m+x_0\leqslant b_{\gamma,x_0}$. Therefore we shall choose

$$d\leqslant\frac{1}{\Delta_m}(b_{\gamma,x_0}-x_0). \tag{2.9}$$

Let us now deal similarly with a lower-bound of ψ_-^L. We define

$$\mathcal{R}_-^L(t):=\psi_-^L(t;\,t_0,x_0)-x_0.$$

Hence

$$\begin{aligned}\mathcal{R}_-^L(t) &= e^{-\theta(t_0+t)}\left(\psi_-(\rho(t+t_0)-\rho(t_0))+\int_{t_0}^{t_0+t}\beta(s)e^{-\int_0^s\alpha(u)du}ds\right)\\ &\quad+x_0\left(e^{\int_{t_0}^{t+t_0}\alpha(u)du}-1\right)\\ &\geqslant e^{-\theta(t_0+t)}\left(-\frac{d}{\sqrt{e}}+\int_{t_0}^{t_0+t}(\beta(s)+x_0\,\alpha(s))\;e^{-\int_0^s\alpha(u)du}ds\right)\\ &\geqslant e^{-\theta(t_0+t)}\left(-\frac{d}{\sqrt{e}}-\int_{t_0}^{t_0+t}|\beta(s)+x_0\,\alpha(s)|e^{-\int_0^s\alpha(u)du}ds\right)\\ &\geqslant -e^{-\theta(t_0)}e^{\int_{t_0}^{t_0+m}|\alpha(s)|ds}\left(\frac{d}{\sqrt{e}}+\int_{t_0}^{t_0+t}|\beta(s)+x_0\,\alpha(s)|e^{-\int_0^s\alpha(u)du}ds\right).\end{aligned}$$

Using then the same arguments as for the upper bound, we obtain

$$\psi_-^L(t;\,t_0,x_0)\geqslant-\Delta_m d+x_0.$$

The condition $-\Delta_m d+x_0\geqslant a_{\gamma,x_0}$ is equivalent to

$$d\leqslant\frac{1}{\Delta_m}(x_0-a_{\gamma,x_0}). \tag{2.10}$$

Combining (2.9), (2.10) and $d^2\leqslant\rho(t_0+m)-\rho(t_0)$, we deduce the announced statement. □

2.3 WOMS algorithm for L-class diffusions

Let us present now the random walk on spheroids which permits to approximate the L-class diffusion exit time.

ALGORITHM$_m$ (L-class WOMS)

Step 1. Initiate $Z = x_0$ and $\mathcal{T}_\epsilon = 0$

Step 2. While $Z \leqslant b - \epsilon$ and $Z \geqslant a + \epsilon$

Step 2.1 Simulate a couple of independent random variables $(\tau^L, \mathcal{B})$ where τ^L denotes the exit time for the diffusion from the spheroid defined by $\psi^L_\pm$ with coefficient $d = d(\mathcal{T}_\epsilon, Z)$ defined in (2.7) and $\mathcal{B}$ is Bernoulli distributed $\mathcal{B}(\frac{1}{2})$. The r.v. $\mathcal{B}$ indicates if the diffusion hits the lower boundary. Due to symmetry properties, its average equals 1/2.

Step 2.2 If $\mathcal{B} = 1$ then set $Z \leftarrow \psi^L_-(\tau^L; \mathcal{T}_\epsilon, Z)$
otherwise set $Z \leftarrow \psi^L_+(\tau^L; \mathcal{T}_\epsilon, Z)$.

Step 2.3 $\mathcal{T}_\epsilon \leftarrow \mathcal{T}_\epsilon + \tau^L$.

Outcome: $\mathcal{T}_\epsilon$ the approximated exit time from the interval $[a, b]$ for the diffusion $(X_t,\, t \geq 0)$.

As usual let us describe the efficiency of the algorithm. This algorithm is particularly efficient since its averaged number of steps is of the order $|\log(\epsilon)|$ and since its outcome $\mathcal{T}_\epsilon$ converges towards the value of the exit time as ϵ tends to 0. We present these two results in details in the following subsections. Even if the statement of these results look like similar to those presented in the Ornstein-Uhlenbeck context (see [7]), the situations are clearly different since here the coefficients - and therefore the size of the spheroids - are time-dependent.
Since the L-class diffusions are non homogeneous, the sequence $(Z_n)_n$ of successive exit positions, appearing in the algorithm, does not define a Markov chain. We need therefore to consider both the successive times and positions (T_n, X_n) in order to deal with a Markov chain. Here T_n stands for the cumulative time:

$$T_n = \sum_{k=1}^{n} \tau^L_k, \quad n \geq 1. \tag{2.11}$$

3 Properties of the algorithm

3.1 Average number of steps

In order to describe precisely the average number of steps in ALGORITHM$_m$, we introduce two crucial additional hypotheses.

Assumption 3.1. *There exist $q' \in [0,1[$ and $q \in [0,1]$, $C_{\tilde{\sigma},\beta} > 0$ and $\underline{\sigma} > 0$ such that*

$$|\alpha(t)| = \mathcal{O}((\ln t)^{q'}), \quad \text{for large values of } t, \tag{3.1}$$

and

$$\underline{\sigma} \leqslant \tilde{\sigma}(t) \leqslant C_{\tilde{\sigma},\beta}\ t^{q/4}, \quad |\beta(t)| \leqslant C_{\tilde{\sigma},\beta}\ t^{q/4}, \quad \text{for } t \text{ large enough.} \tag{3.2}$$

Assumption 3.2. *There exists $\chi_m > 0$ such that, for any t large enough,*

$$\inf_{s\in[t,t+m]} \tilde{\sigma}(s) \geqslant \chi_m \sup_{s\in[t,t+m]} \tilde{\sigma}(s). \tag{3.3}$$

Theorem 3.1. *Let us assume that Assumptions 3.1 and 3.2 are satisfied for a particular parameter $m > 0$. Then for any parameter $\tilde{q} > q$, there exists a constant $C_{\tilde{q}} > 0$ such that N_ϵ, the number of steps observed in* ALGORITHM$_m$ *has the following upper-bound:*

$$\mathbb{E}[N_\epsilon^{1-\tilde{q}}] \leqslant C_{\tilde{q}} |\log(\epsilon)|,$$

for any $\epsilon > 0$ small enough.

In particular, for a L-class diffusion with bounded coefficients, we can prove that $\mathbb{E}[N_\epsilon] \leqslant C_0 |\log(\epsilon)|$, for ϵ small enough.

Let us notice that ALGORITHM$_m$ can be modified in order to approximate the stopping time $\mathcal{T} \wedge T_{\max}$ where $T_{\max}$ is a fixed time horizon. It suffices in such a situation to observe the path skeleton $(T_n, X_n)_{n\geq 0}$ up to the exit from the domain $[0, T_{\max}] \times [a+\epsilon, b-\epsilon]$. The proof of Theorem 3.1 can be adapted to this modified algorithm: there exists a constant $C > 0$ such that the average number of spheroids satisfies

$$\mathbb{E}[N_\epsilon] \leqslant C |\log(\epsilon)|,$$

for any $\epsilon > 0$ small enough. Since this result only concerns the diffusion process on the restricted time interval $[0, T_{\max}]$, we don't need any particular

assumption on the large time behaviour of the coefficients α, β and $\tilde{\sigma}$. Assumption 3.1 and 3.2 are therefore not necessary for the modified algorithm.

We postpone the proof of Theorem 3.1 and present several preliminary results. First we shall focus our attention on a comparison result between the L-class diffusion and a particular autonomous diffusion. Secondly we describe particular solutions of PDEs related to the diffusion generator. Finally we prove Theorem 3.1 using the martingale theory.

A comparison result for SDEs

We introduce two different results: the first one permits to skip the diffusion coefficient in (2.1) and the second one permits to replace the time-dependent drift term by a constant drift.

Proposition 3.2. *Let $(X_t,\ t \geq 0)$ the solution of the SDE (2.1). We define the strictly increasing function γ by*

$$\int_0^{\gamma(t)} \tilde{\sigma}^2(s)ds = t, \quad t \geq 0.$$

Then $Y_t := X_{\gamma(t)}$ satisfies the following SDE

$$dY_t = \Big(\frac{\alpha(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))}\, Y_t + \frac{\beta(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))}\Big)\, dt + dB_t, \quad t \geq 0, \tag{3.4}$$

where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion.

Proof. Using the definition of Y_t, we get

$$\begin{aligned} Y_t = X_{\gamma(t)} &= x + \int_0^{\gamma(t)} \Big(\alpha(s)X_s + \beta(s)\Big)\, ds + \int_0^{\gamma(t)} \tilde{\sigma}(s)dW_s \\ &= x + \int_0^t \Big(\alpha(\gamma(s))X_{\gamma(s)} + \beta(\gamma(s))\Big)\gamma'(s)\, ds + B_t \\ &= x + \int_0^t \Big(\alpha(\gamma(s))Y_s + \beta(\gamma(s))\Big)\gamma'(s)\, ds + B_t \end{aligned}$$

where $B_t = \int_0^{\gamma(t)} \tilde{\sigma}(s)dW_s$ is a standard Brownian motion. □

We obtain the following comparison result, its proof can be found in [20] (Chapter VI).

Proposition 3.3. *Let $T > 0$ and let us define*

$$\mu_T := \inf_{x\in[a,b],\, t\leq\gamma^{-1}(T)} \Big\{ \frac{\alpha(\gamma(t))}{\tilde\sigma^2(\gamma(t))}\, x + \frac{\beta(\gamma(t))}{\tilde\sigma^2(\gamma(t))} \Big\}.$$

Let $(Z_t^T)_{t\geq 0}$ the Brownian motion with drift satisfying

$$Z_t^T = x + \mu_T\, t + B_t, \quad t \geq 0. \tag{3.5}$$

Then (Y_t) the solution of (3.4) with initial condition x satisfies

$$(Z_t^T \leq Y_t \quad a.s., \quad \forall t \leq \gamma^{-1}(T)) \quad \textit{and} \quad (Z^T_{\gamma(t)} \leq X_t \quad a.s. \quad \forall t \leq T).$$

Remark 3.4. *Choosing rather the particular value*

$$\mu_T := \sup_{x\in[a,b],\, t\leq\gamma^{-1}(T)} \Big\{ \frac{\alpha(\gamma(t))}{\tilde\sigma^2(\gamma(t))}\, x + \frac{\beta(\gamma(t))}{\tilde\sigma^2(\gamma(t))} \Big\},$$

leads to ($Z_t^T \geq Y_t$ a.s. for all $t \leqslant \gamma^{-1}(T)$).

An Initial-Boundary Value problem

We consider a value problem which is directly linked to the L-class diffusions: let $F : (\mathbb{R}_+, [a, b]) \to \mathbb{R}$ be the solution of

$$\frac{\partial F}{\partial t} + (\alpha(t)x + \beta(t))\frac{\partial F}{\partial x} + \frac{1}{2}\tilde\sigma(t)^2\frac{\partial^2 F}{\partial x^2} = 0 \tag{3.6}$$

with initial and boundary conditions $F(0, x) = x$, $F(t, a) = a$, $F(t, b) = b$.

It is well-known (see, for instance, [2], Chap.II) that F admits a probabilistic representation. Indeed

$$F(t, x) = \mathbb{E}_x[X_{t\wedge\mathcal{T}}], \quad \forall t \geq 0, \quad \forall x \in [a, b], \tag{3.7}$$

where $(X_t,\, t \geq 0)$ satisfies (2.1) and $\mathcal{T}$ stands for the first exit time from the interval $[a, b]$. Let us just note that we don't need at that moment to assume or verify that the exit time $\mathcal{T}$ is almost surely finite. Obviously $t \wedge \mathcal{T}$ is finite and this fact permits to properly define the probabilistic representation (3.7). However it would be a nonsense to propose a numerical approximation of an infinite stopping time. Hence, for the sake of completeness, we emphasize the importance of both Assumption 3.1 and 3.2 which imply the finiteness of $\mathcal{T}$ as a by-product of Theorem 3.1 and Theorem 3.8 (see Remark 3.9).
Let us now list some useful properties of the function F. Since the functions

α, β and $\tilde{\sigma}$ in (2.1) are Hölder-continuous, we deduce that F, $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$ are also Hölder-continuous (see, for instance, Theorem 9 of Chapter 3 in [6]). A combination of classical arguments permits to prove the following statement (see the working paper [8] for complement computations).

Lemma 3.5. *The function $x \mapsto F(t,x)$ defined in (3.7) is increasing and continuous on the set $[a,b]$. Moreover there exists $\kappa > 0$ such that $\frac{\partial F}{\partial x}(t,x) \geqslant \kappa$, for all $(t,x) \in \mathbb{R}_+ \times [a,b]$.*

Proposition 3.6. *There exists two constants $\kappa_a > 0$ and $\kappa_b > 0$ such that*

$$F(t,x) - a \leqslant \kappa_a(x-a) \text{ and } b - F(t,x) \leqslant \kappa_b(b-x), \tag{3.8}$$

for all $(t,x) \in \mathbb{R}_+ \times [a,b]$.

Proof. Let us recall the probabilistic representation: $F(t,x) = \mathbb{E}[X^x_{t\wedge\mathcal{T}}]$.
We set $T = \gamma(1)$ and consider (Z^T_t) the diffusion introduced in Remark 3.4 with initial condition $Z^T_0 = X^x_0 = x$. We construct a new continuous diffusion process (Z_t) which is equal to (Z^T_t) on the time interval $[0,1]$ and which satisfies the following SDE otherwise:

$$dZ_t = \Big(\frac{\alpha(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))} Z_t + \frac{\beta(\gamma(t))}{\tilde{\sigma}^2(\gamma(t))} \Big) dt + dW_t, \quad t > 1.$$

Extending the comparison result of Remark 3.4, we know that $Z_t \geq X_{\gamma(t)}$ for all $t \geq 0$. Hence

$$F(t,x) - a \leq \mathbb{E}_x[Z_{\gamma^{-1}(t)\wedge\mathcal{T}(Z)} - a].$$

We split the study into two different cases :

- First case: $\gamma^{-1}(t) \leq 1$. The function $f(x) = e^{-2\mu_T x}$ plays an important role since $f(Z_t)$ is a martingale for $t \leq \gamma(1)$. Using twice the Lagrange mean theorem combined with the optional stopping theorem implies

$$\begin{aligned} F(t,x) - a &\leq \eta_1 \mathbb{E}_x\Big[e^{-2\mu_T Z^T_{t\wedge\tau}} - e^{-2\mu_T a}\Big] = \eta_1\Big(e^{-2\mu_T x} - e^{-2\mu_T a}\Big) \\ &\leq \kappa_a(x-a), \end{aligned}$$

  where $\kappa_a = \Big(\sup_{x\in[a,b]} f'(x)\Big)\Big(\inf_{x\in[a,b]} f'(x)\Big)^{-1}$.

- Second case: $\gamma^{-1}(t) > 1$. We decompose F as follows

$$\begin{aligned} F(t,x) - a &\leq \mathbb{E}_x[(Z_{\gamma^{-1}(t)\wedge\mathcal{T}(Z)} - a)1_{\{\mathcal{T}(Z)>1\}}] \\ &\quad + \mathbb{E}_x[(Z_{\gamma^{-1}(t)\wedge\mathcal{T}(Z)} - a)1_{\{\mathcal{T}(Z)\leq 1\}}] \\ &\leq (b-a)\mathbb{P}_x(\mathcal{T}(Z^T) > 1) + \mathbb{E}_x[(Z^T_{1\wedge\mathcal{T}(Z^T)} - a)1_{\{\mathcal{T}(Z^T)\leq 1\}}] \\ &\leq (b-a)\mathbb{E}_x[\mathcal{T}(Z^\mu)] + \mathbb{E}_x[Z^\mu_{1\wedge\mathcal{T}(Z^\mu)}] - a. \end{aligned}$$

The expression $\mathbb{E}_x[Z^T_{1\wedge\mathcal{T}(Z^T)}] - a$ can be bounded using similar arguments (Lagrange's mean and optional stopping theorems) as those presented in the first part of the proof. Moreover, let us note that the function $g(x) := \mathbb{E}_x[\mathcal{T}(Z^T)]$ is solution ([2], page 45, Theorem 1.2) of

$$\frac{1}{2}\,g'' + \mu_T g' = -1 \quad \text{for } x \in]a,b[\quad \text{and } g(a) = g(b) = 0.$$

We recall that $T = \gamma(1)$ and μ_T is defined in Remark 3.4. The explicit solution of this equation is given by

$$g(x) = \frac{(b-a)(e^{-2\mu_T a} - e^{-2\mu_T x})}{\mu_T(e^{-2\mu_T a} - e^{-2\mu_T b})} - \frac{(x-a)}{\mu_T}.$$

Applying once again Lagrange's mean theorem, we obtain the existence of a constant $C_g > 0$ such that $g(x) \leqslant C_g(x-a)$ for all $x \in [a,b]$. Using similar arguments (just replacing Remark 3.4 by Proposition 3.3), we also prove that $b - F(t,x) \leqslant \kappa_b(b-x)$.

□

Proof of Theorem 3.1.

We already presented all the necessary ingredients in order to prove the statement of Theorem 3.1 which concerns the average number of steps.

Proof. Our choice for the bound of the average number of steps is based on the martingale theory. We recall that F is defined by (3.7) and introduce another important function H defined by $H = V \circ F$ with

$$V(x) = \log\left(\frac{(x-a)(b-x)}{\gamma\epsilon(b-a-\gamma\epsilon)}\right). \tag{3.9}$$

Let us note that V is non negative on the whole interval $[a+\gamma\epsilon, b-\gamma\epsilon]$. Since F is a the solution of (3.6), the function H just introduced satisfies

the following partial differential equation:

$$\frac{\partial H}{\partial t} + (\alpha(t)x + \beta(t))\frac{\partial H}{\partial x} + \frac{1}{2}\tilde{\sigma}(t)^2\frac{\partial^2 H}{\partial x^2} = \frac{1}{2}\tilde{\sigma}(t)^2 V''(F(t,x))\left(\frac{\partial F}{\partial x}(t,x)\right)^2. \tag{3.10}$$

Let us also recall that (T_n, X_n) defined in (2.11) is the sequence of successive exit times and exit positions issued from ALGORITHM_m.
We focus our attention on the sequence $Z_n = H(X_n) + G(n)$ with $G(0) = 0$. Here G stands for a positive function, we are going to precise this function in the sequel. This stochastic process is a super-martingale with respect to the Brownian filtration $(\mathcal{F}_{T_n})_{n\in\mathbb{N}}$. Using Itô's formula and the partial differential equation satisfied by H, we obtain for $\mathcal{D}_n := \mathbb{E}[Z_{n+1} - Z_n|\mathcal{F}_{T_n}]$,

$$\begin{aligned}
\mathcal{D}_n &= \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{\partial H}{\partial t}(s, X_s) + (\alpha(s)X_s + \beta(s))\frac{\partial H}{\partial x}(s, X_s)\right. \\
&\quad \left. + \frac{1}{2}\tilde{\sigma}(s)^2\frac{\partial^2 H}{\partial x^2}(s, X_s)ds \,\middle|\, \mathcal{F}_{T_n}\right] \\
&\quad + \mathbb{E}[M_{n+1} - M_n|\mathcal{F}_{T_n}] + (G(n+1) - G(n)) \\
&= \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{1}{2}\tilde{\sigma}(s)^2 V''(F(s, X_s))\left(\frac{\partial F}{\partial x}(s, X_s)\right)^2 ds \,\middle|\, \mathcal{F}_{T_n}\right] \\
&\quad + (G(n+1) - G(n)),
\end{aligned}$$

where $(M_n)_{n\in\mathbb{N}} = \left(\int_0^{T_n} \tilde{\sigma}(s)\frac{\partial H}{\partial x}(s, X_s)dW_s\right)_{n\in\mathbb{N}}$ is a martingale. Using Lemma 3.5, Proposition 3.6 and the lower bound $\underline{\sigma}$ of $\tilde{\sigma}$ we obtain

$$\mathcal{D}_n \leq -\frac{1}{2}\underline{\sigma}^2\kappa^2(\mathcal{I}(a) + \mathcal{I}(b)) + G(n+1) - G(n), \tag{3.11}$$

where $\mathcal{I}(x) = \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{1}{\kappa_x^2(X_s - x)^2}\, ds \,\middle|\, \mathcal{F}_{T_n}\right]$.
We aim to bound by below the previous integral by considering the shape of the n^{th} spheroid:

$$\begin{aligned}
\psi_+^L(t) - a_{\gamma,X_n} &\leqslant d_n\Delta_m + X_n - a_{\gamma,X_n} \\
&\leqslant \min(1, \kappa_-)(X_n - a_{\gamma,X_n}) + X_n - a_{\gamma,X_n} \\
&\leq 2(X_n - a).
\end{aligned} \tag{3.12}$$

This bound implies

$$\mathcal{I}(a) \geqslant \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{ds}{4\kappa_a^2(X_n-a)^2}\middle|\mathcal{F}_{T_n}\right] = \mathbb{E}\left[\frac{T_{n+1}-T_n}{4\kappa_a^2(X_n-a)^2}\middle|\mathcal{F}_{T_n}\right]$$
$$= \mathbb{E}\left[\frac{\rho_L^{-1}(\rho_L(T_n)+\tau_{n+1})}{4\kappa_a^2(X_n-a)^2}\middle|\mathcal{F}_{T_n}\right]$$

where τ_{n+1} is the Brownian exit time from the spheroid of parameter size d_n.

$$\mathcal{I}(a) \geq \mathbb{E}\left[\frac{\tau_{n+1}}{4\kappa_a^2 r_n(X_n-a)^2}\middle|\mathcal{F}_{T_n}\right]$$

where r_n is the maximum of the derivative ρ' on the time interval $[T_n, T_n+m]$ which contains $[T_n, \rho_L^{-1}(\rho_L(T_n)+\tau_{n+1})]$. We note that $\tau_{n+1} \sim d_n^2\tau$ where τ denotes the Brownian exit time from the Brownian spheroid of parameter 1. Hence

$$\mathcal{I}(a) \geq \frac{d_n^2}{4\kappa_a^2 r_n(X_n-a)^2}\ \mathbb{E}[\tau].$$

Similarly to (3.12) we have $b_{\gamma,X_n} - \psi_-^L(t) \leq 2(b-X_n)$ and the same arguments just presented lead to

$$\mathcal{I}(b) = \mathbb{E}\left[\int_{T_n}^{T_{n+1}} \frac{ds}{\kappa_b^2(b-X_s)^2}\middle|\mathcal{F}_{T_n}\right] \geqslant \frac{d_n^2}{4\kappa_b^2 r_n(b-X_n)^2}\ \mathbb{E}[\tau].$$

Setting $\kappa_{ab} = \max(\kappa_a, \kappa_b)$, we obtain

$$\mathcal{D}_n \leqslant -\frac{d_n^2}{r_n\kappa_{ab}^2}\ \mathbb{E}[\tau]\left(\frac{1}{(b-X_n)^2}+\frac{1}{(X_n-a)^2}\right) + G(n+1) - G(n).$$

Let us first consider the case: $X_n - a \leqslant b - X_n$ (the other case can be studied in a similar way, it suffices to replace $X_n - a_{\gamma,X_n}$ by $b_{\gamma,X_n} - X_n$). Then $d_n = \frac{\min(1,\kappa_-)}{\Delta_m}(X_n - a_{\gamma,X_n})$ and

$$\mathcal{D}_n \leqslant -2\frac{d_n^2}{r_n\kappa_{ab}^2}\ \mathbb{E}[\tau]\frac{1}{(X_n-a)^2} + G(n+1) - G(n)$$
$$\leqslant -\frac{\min(1,\kappa_-)^2}{r_n\Delta_m^2\kappa_{ab}^2}\ \mathbb{E}[\tau] + G(n+1) - G(n).$$

We finally find G by seeking a lower bound of $\frac{\min(1,\kappa_-)^2}{r_n\Delta_m^2}$. We consider two different cases:

First case: $\kappa_- \geq 1$. We introduce α_n, β_n and $\tilde{\sigma}_n$ the maximum of $|\alpha|$ respectively $|\beta|$ and $\tilde{\sigma}$ on the time interval $[0, nm]$. The definition of Δ_m given by (2.8) and the definition of ρ by (2.4) lead to

$$\begin{aligned}\Delta_m^2 r_n &\leqslant e^{4\int_{T_n}^{T_n+m}|\alpha(s)|\,ds}\tilde{\sigma}_n^2\left(\frac{1}{\sqrt{e}}+\sqrt{\int_{T_n}^{T_n+m}\frac{|\beta(s)|^2}{\tilde{\sigma}(s)^2}ds}\right)^2\\ &\leqslant e^{4m\alpha_n}\tilde{\sigma}_n^2\left(\frac{1}{\sqrt{e}}+\sqrt{m}\frac{\beta_n}{\underline{\sigma}}\right)^2.\end{aligned}$$

For the other case: $\kappa_- < 1$

$$\frac{\min(1,\kappa_-)^2}{r_n\Delta_m^2} \geqslant \frac{\rho(T_n+m)-\rho(T_n)}{r_n(b-a)^2} = \frac{\int_0^m \rho'(T_n+s)ds}{r_n(b-a)^2}.$$

Using the definitions of ρ, r_n and the continuity of $\tilde{\sigma}$, there exists $t_0 \in [T_n, T_n+m]$ such that $r_n = \rho'(t_0)$ and therefore

$$\begin{aligned}\frac{\rho'(T_n+s)}{r_n} &= \frac{\tilde{\sigma}^2(T_n+s)}{\tilde{\sigma}^2(t_0)}\,e^{-2\int_{t_0}^{T_n+s}\alpha(u)\,du} \geq \frac{\tilde{\sigma}^2(T_n+s)}{\tilde{\sigma}^2(t_0)}\,e^{-2|T_n+s-t_0|\alpha_n}\\ &\geq \frac{\tilde{\sigma}^2(T_n+s)}{\tilde{\sigma}^2(t_0)}\,e^{-2m\alpha_n}.\end{aligned}$$

Since $\tilde{\sigma}$ satisfies Assumption 3.2, we obtain the following lower bound by integrating with respect to the variable s,

$$\frac{\min(1,\kappa_-)^2}{r_n\Delta_m^2} \geq \frac{m\chi_m}{(b-a)^2}e^{-2m\alpha_n}.$$

Denoting ζ_{n+1} the minimum of the two quantities previously computed, we define recursively the sequence $G(n)$ by

$$G(n+1)-G(n)=\zeta_{n+1}, \quad \forall n \geqslant 0, \quad \text{and } G(0)=0.$$

The sum of these increments leads to

$$\sum_{i=0}^{n-1} G(i+1)-G(i) = \sum_{i=1}^{n}\zeta_i = G(n)-G(0) = G(n).$$

For any parameter $\tilde{q} > q$, Assumption 3.1 implies the existence of a constant $\tilde{C} > 0$ independent of ϵ such that

$$G(n) \geqslant \frac{1}{\tilde{C}}\sum_{k=1}^{n}\frac{1}{k^{\tilde{q}}} \geqslant \frac{1}{\tilde{C}(1-\tilde{q})}(n^{1-\tilde{q}}-1), \quad \forall n \geq 1. \tag{3.13}$$

Moreover the particular choice of the function G permits to obtain $\mathcal{D}_n \leq 0$ for all n. Consequently $Z_n = H(n, X_n) + G(n)$ is a super-martingale. A generalization of Proposition A.1 permits to obtain the upper bound

$$\mathbb{E}[G(N_\epsilon)] \leq H(0, x_0) = V \circ F(0, x_0) = V(x_0). \tag{3.14}$$

Combining (3.13), (3.14) and the definition of the function V in (3.9) leads to

$$\mathbb{E}[N_\epsilon^{1-\tilde{q}}] \leqslant \tilde{C}(1-\tilde{q}) \log\left(\frac{(x_0-a)(b-x_0)}{\gamma\epsilon(b-a-\gamma\epsilon)}\right) + 1.$$

This bound corresponds to the announced result. In order to conclude the proof, we just need to precise that N_ϵ is a.s. finite, see Lemma 3.7. Such a condition is required to apply the generalization of Proposition A.1. □

Lemma 3.7. *The stopping procedure N_ϵ of* ALGORITHM$_m$ *is a.s. finite. Moreover the outcome of the algorithm $\mathcal{T}_\epsilon$ is stochastically upper bounded by $\mathcal{T}$, the diffusion first exit time.*

Proof. Step 1. We emphasize a link between a sample of a L-class diffusion process and the Markov chain generated by the algorithm, denoted $((T_n, X_n))_{n\in\mathbb{N}}$ with $(T_0, X_0) = (0, 0)$.
Let us consider a sample of a L-class diffusion. At the starting point of this path, we create a spheroid of maximal size which belongs to the set $[a, b] \times \mathbb{R}_+$. The first intersection point of this spheroid and the path gives us a first point (t_1, z_1). This construction implies that (t_1, z_1) and (T_1, X_1) are identically distributed. Then considering (t_1, z_1) as a new starting point we construct a spheroid of maximal size and denote by (t_2, z_2) the first intersection point between this new spheroid and the diffusion path starting in (t_1, z_1). Once again we get by construction that (t_2, z_2) and (T_2, X_2) are identically distributed. We build step by step a sequence $((t_n, z_n))_{n\in\mathbb{N}}$ of intersections between the considered sample and the spheroids in such a way that the sequences $((t_n, z_n))_{n\geq 0}$ and $((T_n, X_n))_{n\geq 0}$ are identically distributed.
If we introduce N_ϵ the stopping time appearing in the stopping procedure of the algorithm and $\tilde{N}_\epsilon = \inf\{n \in \mathbb{N}, z_n \notin [a+\epsilon, b-\epsilon]\}$, the identity in law of those random variables holds. By construction, $t_n \leqslant T$ for all $n \in \mathbb{N}$, where T stands for the diffusion first exit time from the interval $[a, b]$. This inequality remains true when t_n is replaced by the random stopping time $t_{\tilde{N}_\epsilon}$.
Since $t_{\tilde{N}_\epsilon}$ and t_{N_ϵ} are identically distributed, we deduce that the outcome of

ALGORITHM$_m$ is stochastically smaller than $\mathcal{T}$.
Step 2. We prove now that N_ϵ is a.s. finite. Using (2.11) and (2.6) we obtain

$$T_n = \rho_L^{-1}(d_1^2\tau_1 + d_2^2\tau_2 + \ldots + d_n^2\tau_n),$$

where $(\tau_k)_{k\geqslant 1}$ is a sequence of independent Brownian exit times from the unit spheroid and d_k represents the size of the spheroid (2.7) starting in (T_k, X_k) and included in $[a, b]$. Let $t_0 > 0$. Then

$$\begin{aligned}\mathbb{P}(T_n \leqslant t_0) &= \mathbb{P}(d_1^2\tau_1 + d_2^2\tau_2 + \ldots + d_n^2\tau_n \leqslant \rho_L^{-1}(t_0)) \\ &\leqslant \mathbb{P}\left(\tau_1 + \tau_2 + \ldots + \tau_n \leqslant \frac{\rho_L^{-1}(t_0)}{\underline{d}(t_0)}\right),\end{aligned}$$

where $\underline{d}(t_0)$ is defined by

$$\underline{d}(t_0) = \inf_{x\in[a+\epsilon,b-\epsilon],\ t\leqslant t_0} d(x,t) > 0.$$

Since $\displaystyle\sum_{k+1}^{n} \tau_k$ tends to $+\infty$ a.s.,

$$\lim_{n\to+\infty} \mathbb{P}(T_n \leqslant t_0) = \mathbb{P}(T_\infty \leqslant t_0) = 0, \quad \forall t_0 > 0.$$

We deduce that $\lim_{n\to+\infty} T_n = +\infty$ a.s. Combining this limiting result to the first step of the proof, that is $T_n \overset{(d)}{\leqslant} T$, implies: $N_\epsilon < +\infty$ a.s. □

3.2 Bounds for the exit time distribution

The second important result in the study of the algorithm is the description of the convergence. It is of prime interest to known how close the outcome of the algorithm and the exit time of the L-class diffusion are. The convergence result is essentially based on the strong relation between the Brownian motion and the L-class diffusion.

Theorem 3.8. *Let us denote by $\overline{\alpha}_t$ (respectively $\overline{\beta}_t$) the maximal value of the function $|\alpha|$ (resp. $|\beta|$) on the interval $[0, t]$. We also introduce F the cumulative distribution function of the L-class diffusion exit time from the interval $[a, b]$ and F_ϵ the distribution function of the algorithm outcome. Then, for any $t \geq 0$ and any $\rho > 1$ there exists $\epsilon_0 > 0$ such that*

$$\left(1 - \rho\sqrt{\epsilon}\,\frac{1+\overline{\beta}_t}{\underline{\sigma}}\right) F_\epsilon(t-\epsilon) \leqslant F(t) \leqslant F_\epsilon(t), \quad \forall \epsilon \leq \epsilon_0, \tag{3.15}$$

the constant $\underline{\sigma}$ being defined in (3.2)*. Moreover this convergence is uniform on each compact subset of the time axis.*

Remark 3.9. *The combination of both Theorem 3.1 and Theorem 3.8 points the finiteness of the diffusion exit time out. The first statement ensures that* ALGORITHM$_m$ *requires a finite number of iterations almost surely (the average number being finite). The second result explains how close the exit time of the diffusion and the algorithm outcome are. In particular, as an immediate consequence of* (3.15)*,* $\lim_{t\to\infty} F(t) = 1$*. Of course, such a crucial property is strongly related to the fact that the diffusion generator is uniformly parabolic, see the condition* (3.2)*.*

Proof. As in Lemma 3.7, we build step by step a sequence $((t_n, z_n))_{n\in\mathbb{N}}$ of intersections between the path of the L-class diffusion process and the spheroids in such a way that the sequences $((t_n, z_n))_{n\geq 0}$ and $((T_n, X_n))_{n\geq 0}$ are identically distributed.
If we introduce N_ϵ the stopping time appearing in the stopping procedure of the algorithm and $\tilde{N}_\epsilon = \inf\{n \in \mathbb{N}, z_n \notin [a+\epsilon, b-\epsilon]\}$, the identity in law of those random variables holds. By construction, $t_n \leqslant \mathcal{T}$ for all $n \in \mathbb{N}$, where $\mathcal{T}$ stands for the diffusion first exit time from the interval $[a, b]$. This inequality remains true when t_n is replaced by the random stopping time $t_{\tilde{N}_\epsilon}$. Hence

$$\begin{aligned} 1 - F(t) = \mathbb{P}(\mathcal{T} > t) &= \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t-\delta) + \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} > t-\delta) \\ &\leqslant \mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t-\delta) + 1 - F_\epsilon(t-\delta), \quad \forall t \geq 0. \end{aligned} \tag{3.16}$$

We focus our attention on the first term of the r.h.s. Using the strong Markov property, we obtain

$$\mathbb{P}(\mathcal{T} > t, t_{\tilde{N}_\epsilon} \leqslant t-\delta) \leqslant F_\epsilon(t-\delta) \sup_{(y,s)\in([a,a+\epsilon]\cup[b-\epsilon,b])\times[0,t-\delta]} \mathbb{P}_{(y,\tau)}(\mathcal{T} > \delta). \tag{3.17}$$

Let us consider the case $y \in [b-\epsilon, b]$ (the study of the other case $y \in [a, a+\epsilon]$ is left to the reader since it suffices by symmetry to use exactly the same arguments). We first note that, for any $y \in [b-\epsilon, b]$,

$$\mathbb{P}_{(y,s)}(\mathcal{T} > \delta) \leq \mathbb{P}_{(y,s)}(\mathcal{T}_b > \delta) \leq \mathbb{P}_{(b-\epsilon,s)}(\mathcal{T}_b > \delta),$$

where $\mathcal{T}_b$ stands for the first passage time through the level b. Let us introduce several notations: we denote the translated function $\alpha_s(t) := \alpha(s+t)$ (similar definitions for $\tilde{\sigma}_s$, β_s and ρ_s are defined by using the translated functions in (2.4)). The diffusion process on the time interval $[s, s+\delta]$ can be

expressed using these translated functions. The condition $\mathcal{T}_b > \delta$ is equivalent to $\sup_{0\leq r\leq\delta} X_{s+r} < b$ and becomes, for all $r \leq \delta$,

$$b - \epsilon + e^{2\int_0^r \alpha_s(u)\,du} W_{\rho_s(r)} + e^{\int_0^r \alpha_s(u)\,du} \int_0^r \beta_s(u) e^{-\int_0^u \alpha_s(w)\,dw}\,du < b. \quad (3.18)$$

Since $s \in [0, t-\delta]$ and $r \leq \delta$, we obtain the following bound:

$$\rho_s(\delta) \geq \underline{\sigma}^2\, \frac{1 - e^{-2\overline{\alpha}_t\delta}}{2\overline{\alpha}_t}.$$

The inequality (3.18) implies

$$\frac{1}{\sqrt{\rho_s(\delta)}} \sup_{0\leq r\leq\delta} W_{\rho_s(r)} \leq \frac{e^{2\overline{\alpha}_t\delta}}{\sqrt{1 - e^{-2\overline{\alpha}_t\delta}}} \frac{\sqrt{2\overline{\alpha}_t}}{\underline{\sigma}} (\epsilon + \overline{\beta}_t\delta) \leq e^{3\overline{\alpha}_t\delta}\, \frac{\epsilon + \overline{\beta}_t\delta}{\underline{\sigma}\sqrt{\delta}}.$$

The Désiré André reflexion principle for the Brownian motion implies that the l.h.s of the previous inequality has the same distribution than the absolute value of a standard gaussian random variable: $|G|$. Hence, for any $y \in [b-\epsilon, b]$ and for any $s \leq t - \delta$:

$$\mathbb{P}(\mathcal{T}_b > \delta) \leq \mathbb{P}\left(|G| \leq e^{3\overline{\alpha}_t\delta}\, \frac{\epsilon + \overline{\beta}_t\delta}{\underline{\sigma}\sqrt{\delta}} \right) \leq \sqrt{\frac{2}{\pi}}\; e^{3\overline{\alpha}_t\delta}\, \frac{\epsilon + \overline{\beta}_t\delta}{\underline{\sigma}\sqrt{\delta}}. \quad (3.19)$$

It suffices to choose $\delta = \epsilon$ in the previous inequality and to combine with (3.16) in order to prove the statement of the theorem. □

4 Numerical application

In order to illustrate the efficiency of ALGORITHM$_m$, we present numerical results associated to two particular linear diffusions.

Example 1 (periodic functions). Let us consider $(X_t)_{t\geq 0}$ the solution of (2.1) with

$$\alpha(t) = \frac{\cos(t)}{2 + \sin(t)}, \quad \beta(t) = \cos(t), \quad \tilde{\sigma}(t) = 2 + \sin(t).$$

Let us just notice that α satisfies $\alpha(t) = \frac{\tilde{\sigma}'(t)}{\tilde{\sigma}(t)}$, such a property simplifies the link between the diffusion process and a standard one-dimensional Brownian motion. In particular, we obtain a simple expression of the time change

appearing in (2.3): $\rho(t) = 4t$. Indeed (2.4) implies

$$\begin{aligned}\rho(t) &= \int_0^t (2+\sin(s))^2 e^{-2\int_0^s \frac{\cos(u)}{2+\sin(u)}du} ds \\ &= \int_0^t (2+\sin(s))^2 e^{-2(\log(2+\sin(s))-\log(2))} ds = 4t.\end{aligned}$$

Using Proposition 2.3, we determine the frontiers of the typical spheroid used in ALGORITHM$_m$.

Proposition 4.1. *If we denote by $\psi_\pm^L(t; t_0, X_{t_0})$ the spheroid starting in (t_0, X_{t_0}), we obtain*

$$\begin{aligned}\psi_\pm^L(t; t_0, X_{t_0}) :=& \frac{2+\sin(t+t_0)}{2}\Big(\psi_\pm(4t) + 2\log\Big(\frac{2+\sin(t+t_0)}{2+\sin(t_0)}\Big)\Big) \\ &+ \Big(\frac{2+\sin(t+t_0)}{2+\sin(t_0)}\Big) X_{t_0}. \end{aligned} \tag{4.1}$$

and the exit time $\tau^{t_0} = \inf\{t > 0 : \ X_t \notin [\psi_-^L(t; t_0, X_{t_0}), \psi_+^L(t; t_0, X_{t_0})]\}$ satisfies

$$\tau^{t_0} \overset{d}{=} \frac{1}{4}\tau \tag{4.2}$$

where $\tau = \inf\{t > 0 : \ W_t \notin [\psi_-(t), \psi_+(t)]\}$.

The random walk on spheroids is therefore built using the typical boundaries (4.1). At each step of the algorithm, we need to use a scale parameter d in order to shrink or enlarge the spheroid size in such a way that the domains always stay in the interval $[a, b]$. The general statement concerning the scale parameter (2.7) can be improved for this particular example.

Let $m > 0$ and $0 < \gamma < 1$. We recall that a_{γ,x_0} and b_{γ,x_0} are defined by $a_{\gamma,x} = a + \gamma(x-a)$ and $b_{\gamma,x} = b - \gamma(b-x)$. We choose the scale parameter d in such a way that it satisfies

$$d = \begin{cases} \frac{\min(1,\kappa_+)}{\Delta_m}(b_{\gamma,x_0} - x_0) \text{ if } b - x_0 \leqslant x_0 - a \\ \frac{\min(1,\kappa_-)}{\Delta_m}(x_0 - a_{\gamma,x_0}) \text{ if } x_0 - a \leqslant b - x_0 \end{cases}$$

with

$$\Delta_m = \frac{3}{2}\left(\frac{1}{\sqrt{e}} + (1 + \max(|a|, |b|))\sqrt{m}\right)$$

and $\kappa_\pm$ are defined by the following equations:

$$\kappa_+(b_{\gamma,x_0} - x_0) = 2\Delta_m\sqrt{m} \text{ and } \kappa_-(x_0 - a_{\gamma,x_0}) = 2\Delta_m\sqrt{m}.$$

We just note that this particular value Δ_m is an easy upper-bound of the parameter emphasized in (2.8). We just adapted the choice of the parameters to the particular diffusion studied in this section. Even if the procedure is close to the method presented in Proposition 2.4, we notice that such a particular choice of Δ_m permits to point out a specific value m such that both $\min(1, \kappa_-)$ and $\min(1, \kappa_+)$ are equal to 1. This value corresponds to

$$m = \left(\frac{\sqrt{\frac{1}{e} + \frac{4}{3}(b-a)(1+\max(|a|,|b|)} - \frac{1}{\sqrt{e}}}{2(1+\max(|a|,|b|))} \right)^2 .$$

Using ALGORITHM$_m$ as in Section 2.3 permits to approximate the first diffusion exit time from the interval $[a, b]$, see Figure 1 and Figure 2.

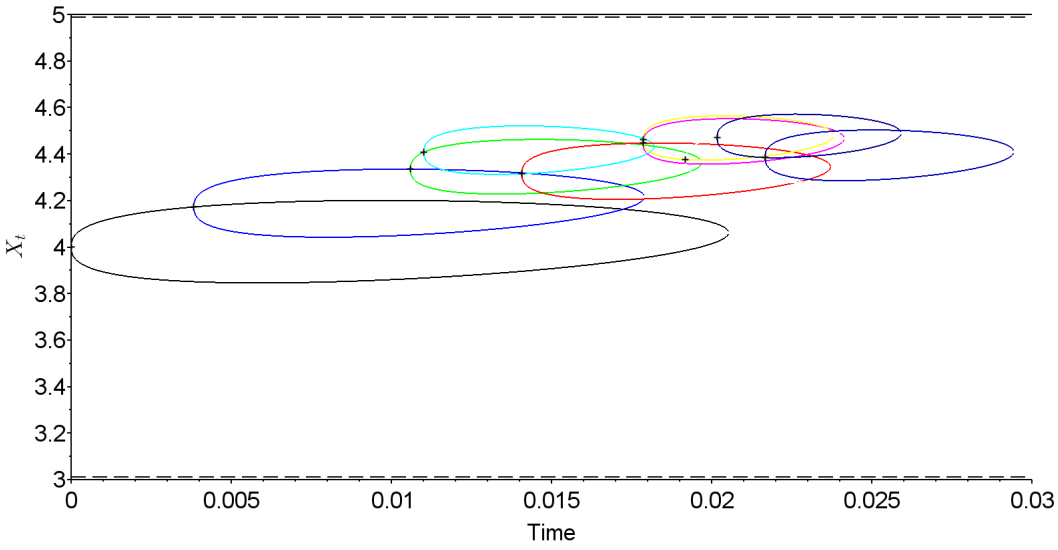

Figure 1: A sample of ALGORITHM$_m$ for the diffusion process starting at $x = 4$ in the interval $[3, 5]$ with $\epsilon = 10^{-2}$ and $\gamma = 10^{-4}$.

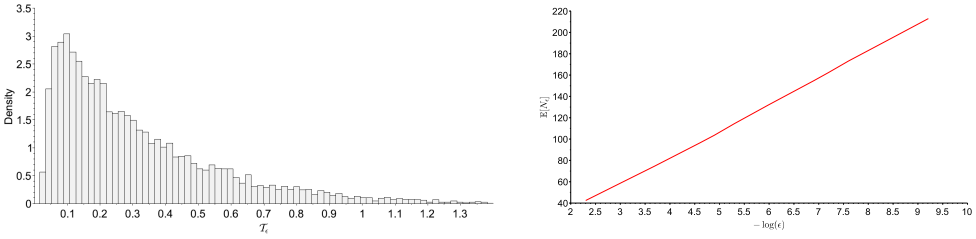

Figure 2: Histogram of the outcome variable for the diffusion (4) with $X_0 = 1$, $[a, b] = [-1, 2]$, $\epsilon = 10^{-2}$ and $\gamma = 10^{-4}$ (left). Average number of steps in ALGORITHM$_m$ for the exit time of $[-1, 2]$ (right, in logarithmic scale).

The CPU efficiency of such an algorithm shall be compared to the efficiency of classical approaches in the exit time approximation framework. We focus on an improved Euler method based on the correction by means of the sharp large deviations estimate of the exit probability (see the procedure described in [1]). We observe the linear diffusion with periodic coefficients starting in $X_0 = 1$ until it exits from the interval $[-1, 2]$. The generation of 100 000 samples of this exit time requires 659 seconds for the improved Euler method (with the step size 10^{-4}) whereas the corresponding generation using the WOMS algorithm requires about 39 seconds for the corresponding choice $\epsilon = 10^{-4}$ (here $\gamma = 10^{-4}$).

Example 2 (polynomial decrease). Let us introduce a diffusion with a polynomial decrease of the mean reversion. We consider (2.1) with

$$\alpha(t) = \frac{1}{2}\frac{1}{1+t}, \quad \beta(t) = 0, \quad \tilde{\sigma}(t) = \sigma_0.$$

Both the time-change function appearing in (2.3) and the typical spheroid frontiers can be explicitly computed. We obtain: $\rho(t) = \sigma_0^2 \log(1+t)$ and the following result due to Proposition 2.3.

Proposition 4.2. *If we denote by $\psi_\pm^L(t; t_0, X_{t_0})$ the spheroid starting in (t_0, X_{t_0}), we have*

$$\begin{aligned}\psi_\pm^L(t; t_0, X_{t_0}) :=& \sqrt{1+t_0+t}\,\psi_\pm(\sigma_0^2(\log(1+t_0+t) - \log(1+t_0))) \\ &+ \frac{\sqrt{1+t_0+t}}{\sqrt{1+t_0}} X_{t_0}. \end{aligned} \tag{4.3}$$

and the exit time $\tau^{t_0} = \inf\{t > 0 : \ X_t \notin [\psi_-^L(t; t_0, X_{t_0}), \psi_+^L(t; t_0, X_{t_0})]\}$ satisfies

$$\tau^{t_0} \stackrel{d}{=} \rho^{-1}(\tau)(t_0 + 1) \tag{4.4}$$

where $\tau = \inf\{t > 0 : \ W_t \notin [\psi_-(t), \psi_+(t)]\}$ *and* $\rho^{-1}(t) = \exp\left(\frac{t}{\sigma_0^2}\right) - 1$.

These particular boundaries (4.3) are the basic components of the algorithm. Of course we need to adjust at each step the size of the spheroid in order to stay in the interval under consideration. The scale parameter d is defined in (2.7) and depends on a fixed arbitrary parameter $m > 0$. In Example 1, the parameter m was optimized in order to reduce the CPU time. Here it is not an easy task to choose a suitable value of m. The algorithm converges in Example 2 whatever the value of m (see Assumption 3.2), that is why we set $m = 1$ for the numerical illustration. The generation of 10 000 samples using the improved Euler method requires 568 seconds (with steps of size 10^{-4}) while it takes only 16 seconds with ALGORITHM$_m$ (with the corresponding choice $\epsilon = 10^{-4}$, $\gamma = 10^{-4}$ and $m = 1$).

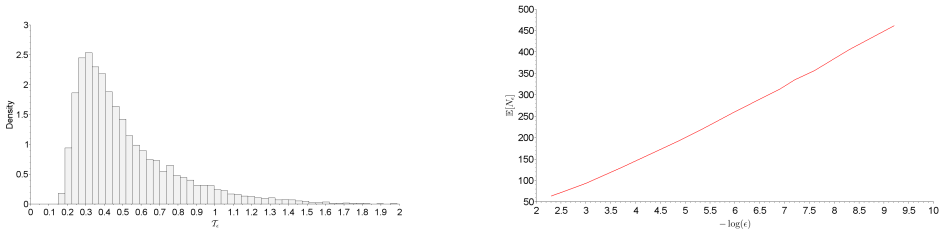

Figure 3: Histogram of the outcome variable for the diffusion (4) with $X_0 = 4$, $[a, b] = [3, 5]$, $m = 1$, $\sigma = 2$, $\epsilon = 10^{-4}$ and $\gamma = 10^{-4}$ (left). Average number of steps in ALGORITHM$_m$ for the exit time of $[3, 5]$ with $m = 1$ and $\sigma_0 = 2$ (right, in logarithmic scale).

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A Potential theory and Markov chains

We introduce a result coming from the potential theory and using Markov chains.
Let us consider a Markov chain $(X_n)_{n\in\mathbb{N}}$ defined on a state space I decomposed into two distinct subsets K and ∂K, ∂K being the so-called frontier. Let us define $N = \inf\{n \in \mathbb{N}, X_n \in \partial K\}$ the hitting time of ∂K. We assume that N is a.s. finite, then the following statement holds:

Proposition A.1. *Let G be a positive increasing function. If there exists a function U such that the sequence $(H(n \wedge N, X_{n\wedge N}))_{n\in\mathbb{N}}$ is non negative*

and if the sequence $(H(n \wedge N, X_{n\wedge N}) + G(n \wedge N))_{n\in\mathbb{N}}$ *represents a supermartingale adapted to the natural filtration of the considered Markov chain* (X_n)*, then*

$$\mathbb{E}_x[G(N)] \leqslant H(0, x), \quad \forall x \in K.$$

The proof of this classical upper-bound is left to the reader, it is essentially based on the optimal stopping theorem and on the monotone convergence theorem (see, for instance, [13], p139).

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