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Abstract

The influence of a time-periodic forcing on stochastic processes can essentially be emphasized in the large time behaviour of their paths. The statistics of transition in a simple Markov chain model permits to quantify this influence. In particular a functional Central Limit Theorem can be proven for the number of transitions between two states chosen in the whole finite state space of the Markov chain. An application to the stochastic resonance is presented.

Key words and phrases: Markov chain, Floquet multipliers, central limit theorem, large time asymptotic, stochastic resonance.

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Introduction

The description of natural phenomenon sometimes requires to introduce stochastic models with periodic forcing. The simplest model used to interpret for instance the abrupt changes between cold and warm ages in paleoclimatic data is a one-dimensional diffusion process with time-periodic drift [6]. This periodic forcing is directly related to the variation of the solar constant (Milankovitch cycles). In the neuroscience framework, such periodic forced model is also of prime importance: the firing of a single neuron stimulated by a periodic input signal can be represented by the first passage time of a periodically driven Ornstein-Uhlenbeck process [19] or other extended models [14]. Moreover let us note that seasonal autoregressive moving average models have been introduced in order to analyse and forecast statistical time series with periodic forcing. Recently the time dependence of the volatility in financial time series leaded to emphasize periodic autoregressive conditional heteroscedastic models. Whereas several statistical models permit to deal with time series, the influence of periodic forcing on time-continuous stochastic processes concerns only few mathematical studies. Let us note a nice reference in the physics literature dealing with this research subject [13].

Therefore we propose to study a simple Markov chain model evolving in a time-periodic environment (already introduced in the stochastic resonance context [12] and [9]) and in particular to focus our attention to its large time asymptotic behaviour. Since the dynamics of the Markov chain is not time-homogeneous, the classical convergence towards the invariant measure and the related convergence rate cannot be used.

Description of the model. Let us consider a time-continuous irreducible Markov chain evolving in the state space $\mathcal{S} = \{s_1, s_2, \ldots, s_d\}$ with $d \geq 2$. The transition rate from state s_i to state s_j is denoted by $\varphi^0_{i,j}$, for $i \neq j$. We assume that $\varphi^0_{i,j} \geq c$ for some positive constant c and for any $i \neq j$. We perturb this initial process by a periodic forcing of period T; it means that the transition rates $\varphi^0_{i,j}$ are increased using additional non negative periodic functions $\varphi^p_{i,j}$. The obtained Markov chain is denoted by $(X_t)_{t\geq 0}$ and its infinitesimal generator is given by

$$Q_t = \begin{pmatrix} -\varphi_{1,1}(t) & \varphi_{2,1}(t) & \ldots & \varphi_{d,1}(t) \\ \varphi_{1,2}(t) & -\varphi_{2,2}(t) & & \varphi_{d,2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{1,d}(t) & \varphi_{2,d}(t) & \ldots & -\varphi_{d,d}(t) \end{pmatrix}, \tag{0.1}$$

Here $\varphi_{i,j} = \varphi^0_{i,j} + \varphi^p_{i,j}$ are T-periodic functions representing the transition rate from state s_i to s_j. In particular, the transitions rates satisfy:

$$\varphi_{i,j}(t) \geq c > 0 \quad \text{for any } (i,j) \in \mathcal{S}^2. \tag{H}$$

We also assume that $\varphi_{i,j}$ are càdlàg functions.

In order to describe precisely the paths of the chain (X_t), we define transitions statistics: $\mathcal{N}^{i,j}_t$ corresponds to the number of switching from state s_i to s_j up to time t. For notational convenience, we focus our attention on $\mathcal{N}_t := \mathcal{N}^{1,2}_t$. Obviously knowing the processes $(\mathcal{N}^{i,j}_t)$ for any $1 \leq i, j \leq d$ is equivalent to know the behaviour of (X_t).

Main result. Let us first note that, in the higher dimensional space $[0,T]\times\mathcal{S}$, we can define a Markov process $(t \bmod T, X_t)_{t\geq 0}$ which is time-homogeneous and admits a unique invariant measure $\mu = (\mu_i(t))_{1\leq i\leq d,\, t\in[0,T[}$. The main results can then be stated. The periodic forcing implies that the distribution of the Markov chain (X_t) converges as time elapses toward the unique invariant measure μ (the sense of this convergence is made precise in Section 1). Moreover the first moments of the statistics $\mathcal{N}_t$ satisfy:

$$\lim_{t\to\infty} \frac{1}{t}\,\mathbb{E}[\mathcal{N}_t] = \frac{1}{T}\int_0^T \varphi_{1,2}(s)\mu_1(s)\,ds,$$

and there exists a constant $\kappa_\varphi > 0$ such that $\lim_{t\to\infty} \mathrm{Var}(\mathcal{N}_t)/t = \kappa_\varphi$. The explicit value of the constant κ_φ is emphasized in Section 2.1. Using these two moment asymptotics, we can prove a Central Limit Theorem: the number of transitions between two given states during n periods is asymptotically gaussian distributed: the process

$$\left(\frac{\mathcal{N}_{ntT} - \mathbb{E}[\mathcal{N}_{ntT}]}{\sqrt{\mathrm{Var}(\mathcal{N}_{ntT})}}\right)_{0\leq t\leq 1}$$

converges in distribution towards the standard Brownian motion as n tends to infinity (Theorem 2.6).

Application. The explicit expression of the mean number of transition between two states before time t permits to deal with particular optimization problems appearing in the stochastic resonance framework (see, for instance, [8]). Let us reduce the study to a 2-state space: $\mathcal{S} = \{s_1, s_2\}$ and to the corresponding Markov chain whose transition rates correspond to $\varphi_{1,2}$ respectively $\varphi_{2,1}$, the exit rate of the state s_1 resp. s_2. Let us consider a family of periodic forcing having all the same period T and being parametrized by a variable ϵ, then it is possible to choose in this family the perturbation which has the most influence on the stochastic process, just by minimizing the following *quality measure*:

$$\mathcal{M}(\epsilon) := \left| \int_0^T \varphi_{1,2}^{\epsilon}(s)\mu_1^{\epsilon}(s)ds - 1 \right|.$$

Indeed this expression intuitively means that the asymptotic number of transitions from state s_1 to state s_2 is close to 1. In Section 3 we shall compare this quality measure (already introduced in [21]) to other measures usually used in the physics literature [12].

1 Periodic stationary measure for Markov chains

Before focusing our attention on the paths behaviour of the Markov chain, we describe, in this preliminary section, the fixed time distribution of the random process and, in particular, analyse the existence of a so-called *periodic stationary probability measure – PSPM* (we shall precise this terminology in the following). The marginal law of the Markov chain $(X_t)_{t\geq 0}$ starting from the initial distribution ν_0 and evolving in the state space $\mathcal{S} = \{s_1, \dots, s_d\}$ is given by

$$\nu_i(t) = \mathbb{P}_{\nu_0}(X_t = s_i), \quad 1 \leq i \leq d.$$

This probability measure $\nu = (\nu_1, \dots, \nu_d)^*$ (the symbol $*$ stands for the transpose) constitutes a solution to the following ODE:

$$\dot{\nu}(t) = Q_t \nu(t) \quad \text{and} \quad \nu(0) = \nu_0, \tag{1.1}$$

where the generator Q_t is defined in (0.1). Let us just note that

$$\mathbb{P}(X_{t+h} = s_j | X_t = s_i) = \varphi_{i,j}(t)h + o(h)$$

for $i \neq j$. Moreover the following relation holds

$$\varphi_{i,i} = \sum_{j=1, j\neq i}^{d} \varphi_{i,j}, \quad \forall 1 \leq i \leq d. \tag{1.2}$$

Floquet's theory dealing with linear differential equation with periodic coefficients can thus be applied. In particular we shall prove that $\nu(t)$ converges exponentially fast towards a periodic solution of (1.1), the convergence rate being related to the Floquet multipliers (see Section 2.4 in [4]).

Definition 1.1. *Any T-periodic solution $\nu(t) = (\nu_1, \dots, \nu_d)^*$ of (1.1) is called a* periodic stationary probability measure – PSPM *iff $\nu_i(t) \geq 0$ for all $i \in \{1, \dots, d\}$ and $\sum_{i=1}^{d} \nu_i(t) = 1$ both for all $t \geq 0$.*

The following statement points out the long time asymptotics of the Markov chain.

Theorem 1.2. *The system* (1.1) *has a unique stationary probability measure* $\mu(t)$ *which is* T*-periodic. For any initial condition* ν_0*, the probability distribution* $\nu(t) := \mathbb{P}_{X_t}$ *converges in the large time limit towards* $\mu(t)$*. More precisely the rate of convergence is given by*

$$\lim_{t\to\infty} \frac{1}{t} \log \|\nu(t) - \mu(t)\| \le \mathrm{Re}(\lambda_2) < 0, \tag{1.3}$$

where λ_2 *is the second Floquet exponent associated to* (1.1) *and* (0.1)*;* $\|\cdot\|$ *stands for the Euclidian norm in* $\mathbb{R}^d$*.*

Proof. **Step 1.** *Existence of the periodic invariant measure.* We consider the distribution of the Markov chain (X_t) starting from state s_i, we obtain obviously a probability measure which is solution of the following ode:

$$\dot{\nu}^i(t) = Q_t \nu^i(t), \quad \nu^i(0) = (\delta_{ij})_{j\in\{1,\ldots,d\}}, \tag{1.4}$$

where δ_{ij} stands for the Kronecker symbol. We deduce that the principal matrix solution of (1.1) is given by

$$M(t) = \begin{pmatrix} \nu_1^1(t) & \nu_1^2(t) & \ldots & \nu_1^d(t) \\ \nu_2^1(t) & \nu_2^2(t) & & \nu_2^d(t) \\ \vdots & \vdots & \ddots & \vdots \\ \nu_d^1(t) & \nu_d^2(t) & \ldots & \nu_d^d(t) \end{pmatrix}$$

since $M(0) = \mathrm{Id}_d$, the identity matrix in $\mathbb{R}^d$. The monodromy matrix $M(T)$ is therefore stochastic and strictly positive: $\nu_j^i(T) > 0$ since Q_t satisfies (H). By the Perron-Frobenius theorem (see chapter 8 in [15]), the largest eigenvalue is simple and equal to 1. Moreover, the associated eigenvector $u = (u_1, \ldots, u_d)^*$ is strictly positive and so we define a probability measure using a normalisation procedure: $\frac{u}{\sum_{i=1}^d u_i}$. Consequently there exists a unique periodic invariant probability measure $\mu(t)$ defined by

$$\mu(t) = \frac{M(t)u}{\sum_{i=1}^d u_i}, \quad t \ge 1.$$

Floquet's theory ensures that $\mu(t)$ is T-periodic.
Step 2. *Convergence.* By the Perron-Frobenius theorem, the eigenvalues of the monodromy matrix $M(T)$, also called *Floquet multipliers*, are $\{r_1, r_2, \ldots, r_s\}$, $s \le d$ with $1 = r_1 > |r_2| \ge |r_3| \ge \ldots \ge |r_s|$ and whose associated multiplicities $n_1, \ldots, n_s$ satisfy $n_1 = 1$ and $\sum_{k=1}^s n_k = d$. Let us decompose the space as follows $\mathbb{R}^d = \mathbb{R}\mu(0) \oplus V$ where $\mu(0)$ is the periodic invariant measure at time $t = 0$ and V is a stable subspace for the linear operator $M(T)$. Since the first eigenvalue r_1 is simple, the spectral radius of $M(T)$ restricted to the subspace V satisfies $\rho(M(T)_{|V}) = |r_2| < 1$. So for any probability distribution ν_0, we get $\nu_0 = \alpha\mu(0) + v$ with $\alpha \in \mathbb{R}$ and $v \in V$. Hence

$$\|M(T)^n \nu_0 - \alpha\mu(0)\| = \left\| \Big(M(T)_{|V} \Big)^n v \right\| \le \left\| \Big(M(T)_{|V} \Big)^n \right\| \cdot \|v\|.$$

Using Gelfand's formula (see, for instance [20], p.70) we obtain the asymptotic result

$$\lim_{n\to\infty} \frac{1}{n} \log \|M(T)^n \nu_0 - \alpha\mu(0)\| \le \log(|r_2|) < 0. \tag{1.5}$$

In particular, since $M(T)^n\nu_0$ is a probability measure, we deduce that $\alpha = 1$. Let us just note that the Floquet multiplier r_2 satisfies $r_2 = e^{\lambda_2 T}$ where λ_2 is the associated *Floquet exponent* defined modulo $2\pi/T$. Consequently

$$\log(|r_2|) = T\ \mathrm{Re}(\lambda_2).$$

Let us now consider any time t, $\nu(t)$ is then a probability measure satisfying

$$\nu(t) = M(t)\nu_0.$$

We define $r(t) \in [0, T[$ by $r(t) = t - \lfloor t/T \rfloor T$ and obtain

$$\begin{aligned} \|\nu(t) - \mu(t)\| &= \|M(t)\nu_0 - M(t)\mu_0\| \\ &= \|M(r(t))\, M(\lfloor t/T \rfloor T)\, \nu_0 - M(r(t))\,\mu_0\| \\ &\le \|M(r(t))\| \left\| M(T)^{\lfloor t/T \rfloor}\nu_0 - \mu_0 \right\|. \end{aligned}$$

By (1.5) and since $M(t)$ is a continuous and T-periodic function (bounded operator), we obtain the announced statement (1.3). □

The particular 2-dimensional case

In this section, we focus our attention on the particular 2-dimensional case. As explained in Theorem 1.2, the distribution of the Markov chain $\nu(t) := \mathbb{P}_{X_t}$ starting from the initial distribution ν_0 and evolving in the state space $\mathcal{S} = \{s_1, s_2\}$ converges exponentially fast to the unique PSPM μ. In dimension 2, we can compute explicitly the probability measure $\nu(t)$ and the convergence rate, applying Floquet's theory. This theory deals with linear differential equation with periodic coefficients (see Section 2.4 in [4]). The following statement points out the long time asymptotics of the Markov chain.

Proposition 1.3. *In the large time limit, the probability distribution ν converges towards the unique PSPM μ defined by $\mu(t) = (\mu_1(t), 1 - \mu_1(t))$ and*

$$\mu_1(t) = \mu_1(0)e^{-\int_0^t (\varphi_{1,2}+\varphi_{2,1})(s)ds} + \int_0^t \varphi_{2,1}(s)\ e^{-\int_s^t (\varphi_{1,2}+\varphi_{2,1})(u)du} ds, \tag{1.6}$$

where

$$\mu_1(0) = \frac{I(\varphi_{2,1})}{I(\varphi_{1,2} + \varphi_{2,1})} \quad \text{and} \quad I(f) = \int_0^T f(t)e^{-\int_t^T (\varphi_{1,2}+\varphi_{2,1})(u)du} dt. \tag{1.7}$$

More precisely, if $\nu(0) \neq \mu(0)$ then

$$\lim_{t\to\infty} \frac{1}{t} \log \|\nu(t) - \mu(t)\| = \lambda_2, \tag{1.8}$$

where λ_2 stands for the second Floquet exponent:

$$\lambda_2 = -\frac{1}{T} \int_0^T (\varphi_{1,2} + \varphi_{2,1})(t)\, dt. \tag{1.9}$$

Remark 1.4. *It is possible to transform $(X_t)_{t\geq 0}$ into a time-homogeneous Markov process just by increasing the space dimension. By this procedure $(\mu(t))_{0\leq t<T}$ becomes the invariant probability measure of $(t \mod T, X_t)_{t\geq 0}$.*

Proof. 1. First we study the existence of a unique PSPM. Let $\mu(t)$ be a probability measure thus $\mu_1(t)+\mu_2(t)=1$. If μ satisfies (1.1) then we obtain, by substitution, the differential equation:

$$\dot{\mu}_1(t) = -\varphi_{1,2}(t)\mu_1(t) + \varphi_{2,1}(t)(1-\mu_1(t)).$$

This equation can be solved using the variation of the parameters. The procedure yields (1.6). The periodicity of the solution requires $\mu_1(T)=\mu_1(0)$ and leads to (1.7).
2. The system (1.1) admits two Floquet multipliers ρ_1 and ρ_2. Since there exists a periodic solution, one of the multipliers (let us say ρ_1) is equal to 1 and we can compute the other one using the relation between the product $\rho_1\rho_2$ and the trace of Q_t:

$$\rho_1\rho_2 = \exp\left(\int_0^T \mathrm{tr}(Q_t)\, dt\right).$$

The explicit expression of the trace leads to (1.9). Let us just note that we can link to both Floquet multipliers ρ_1 and ρ_1 the so-called Floquet exponents λ_1 and λ_2 defined (not uniquely) by

$$\rho_1 = e^{\lambda_1 T} \quad \text{and} \quad \rho_2 = e^{\lambda_2 T}.$$

3. Since the Floquet multipliers are different, each multiplier is associated with a particular solution of (1.1). The multiplier $\rho_1=1$ (i.e. $\lambda_1=0$) corresponds to the PSPM since $\mu(t+T)=\rho_1\mu(t)$ for all $t\in\mathbb{R}_+$. For the Floquet exponent λ_2, we consider $\zeta(t)$ the solution of (1.1) with initial condition $\zeta(0)^*=(-1,1)$. Combining both equations of (1.1), we obtain

$$\begin{cases} \zeta_1(t)+\zeta_2(t)=0 \\ \zeta_1(t)-\zeta_2(t) = -2\exp\left(-\int_0^t(\varphi_{1,2}+\varphi_{2,1})(s)ds\right). \end{cases} \tag{1.10}$$

We deduce

$$\zeta(t)^* = \left(-\exp\left(-\int_0^t(\varphi_{1,2}+\varphi_{2,1})(s)ds\right),\ \exp\left(-\int_0^t(\varphi_{1,2}+\varphi_{2,1})(s)ds\right)\right)$$

and we can easily check that $\zeta(t+T)=\zeta(t)e^{\lambda_2 T}$.
The solution of (1.1) with any initial condition is therefore a linear combination of ζ and μ, the solutions associated with the Floquet multipliers. Writing $\nu(0)$ in the basis $(\mu(0),\zeta(0))$ yields $\nu(t)=\alpha\mu(t)+\beta\zeta(t)$, with $\alpha=\nu_1(0)+\nu_2(0)$ (α is equal to 1 in the particular probability measure case) and

$$\beta = \frac{\nu_1(0)-\nu_2(0)}{2} + \alpha\frac{I(\varphi_{2,1})-I(\varphi_{1,2})}{2I(\varphi_{1,2}+\varphi_{2,1})}.$$

Then, if the initial condition is a probability measure, we obtain (1.8) since

$$\|\nu(t)-\mu(t)\| = \|\beta\zeta(t)\| = \sqrt{2}|\beta|e^{-\int_0^t(\varphi_{1,2}+\varphi_{2,1})(s)ds}.$$

□

2 Statistics of the number of transitions

In this section, we aim to describe the number of transitions $\mathcal{N}_t^{i,j}$, up to time t, between two given states s_i and s_j. This information is of prime interest since computing it for a given path is very simple [17]. Recent studies emphasize how to get the probability distribution of this counting process, even in some more general situations: Markov renewal processes including namely the time-homogeneous Markov chains [3].
Moreover counting the transitions permits to get informations about the transition rates of the Markov chain. In the particular time-homogeneous case, the number of transitions during some large time interval are used for estimation purposes (for continuous-time Markov chains see, for instance, [1] and for time discrete Markov chains [2]).
In general, the large time behaviour of $\mathcal{N}_t^{i,j}$ is directly related to the ergodic theorem, the law of large numbers and finally the Central Limit Theorem (for precise hypotheses concerning these limit theorems, see [16]). Let us just discuss a particular situation: the study of a time-discrete Markov chain $(X_n)_{n\geq 0}$ with values in the state space $\mathcal{S} = \{s_1, \dots, s_d\}$ and with transition probabilities π. Let us denote μ its invariant probability measure. In order to describe the number of transitions, we introduce a new Markov chain by defining $Z_n := (X_{n-1}, X_n)$ for $n \geq 1$, valued in the state space $\mathcal{S}^2$. Its invariant measure is therefore $\tilde{\mu}$ defined by

$$\tilde{\mu}(x, y) := \pi(x, y)\mu(x), \quad (x, y) \in \mathcal{S}^2.$$

In this particular situation, the number of transitions of the chain (X_n) is given by

$$\mathcal{N}_n^{1,2} = \sum_{k=1}^{n} 1_{\{X_{k-1}=s_1,\ X_k=s_2\}} = \sum_{k=1}^{n} 1_{(s_1,s_2)}(Z_k).$$

In other words, it corresponds to the number of visits of the state (s_1, s_2) by the chain $(Z_n)_{n\geq 1}$. Consequently, under suitable conditions, the ergodic theorem can be applied:

$$\lim_{n\to\infty} \frac{\mathcal{N}_n^{1,2}}{n} = \tilde{\mu}(s_1, s_2) \quad \text{almost surely.}$$

The Central Limit Theorem specifies the rate of convergence.

However these arguments cannot be applied directly to the periodic forced Markov chain model associated to the infinitesimal generator (0.1) due to essentially two facts:

- the Markov chain $(X_t)_{t\geq 0}$ is time-inhomogeneous
- the Markov chain is a time-continuous stochastic process.

One way to overcome these difficulties is to combine a discrete time-splitting $(t_n)_{n\geq 0}$ on one hand and an increase of the space dimension on the other hand so that $(t_n \bmod T, X_{t_{n-1}}, X_{t_n})$ becomes homogeneous. This procedure seems to be complicated and we choose to present a quite different approach based on a time-spitting and on a functional Central Limit Theorem for weakly dependent random variables introduced by Herrndorf [11]. This result requires to study the asymtotic behaviour of the first moments of $\mathcal{N}_t^{i,j}$ and a mixing property of

the Markov chain.
Let us also mention that usually the Central Limit Theorem and the associated large deviations could be proven using asymptotic properties of the Laplace transform of $\mathcal{N}_t^{i,j}$. Of course such information is not sufficient for a functional CLT. An overview of the conditions can be found in [5].

2.1 Long time asymptotics for the average and the variance

The general d-dimensional case

Let us focus our attention to the two first moments of $\mathcal{N}_t$, the number of transition between two given states, let us say s_1 and s_2. In a homogeneous continuous time Markov chain, the average and the variance of $\mathcal{N}_t$ grows linearly if the process starts with the stationary distribution. What happens if the Markov chain is not homogeneous and in particular, if the transition probabilities depend periodically on time?

Let us introduce different mathematical quantities which play a crucial role in the asymptotic result.

- Let us denote by $M(t)$ the fundamental solution of (1.1), that is:
$$\dot{M}(t) = Q_t M(t), \quad M(0) = \mathrm{Id}. \tag{2.1}$$
- $\Xi(T)$ represents the Jordan canonical form of $M(T)$. P is the matrix basis of this canonical form: $\Xi(T) = P^{-1}M(T)P$. Moreover we denote for any $t \geq 0$,
$$\Xi(t) = P^{-1}M(t)P. \tag{2.2}$$
- Three additional notations: the vector $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$ and the matrices $\check{\mathrm{Id}}^1_{i,j} = 1_{\{i=j\geq 2\}}$ for $1 \leq i, j \leq d$ and $(B_t)_{i,j} = \varphi_{1,2}(t)\delta_{i,2}\delta_{j,1}$.

Theorem 2.1. Asymptotics of the two first moments
The number of transitions from state s_1 to state s_2, denoted by $\mathcal{N}_t$, satisfies the following asymptotic properties.
1. First moment. *For any initial distribution $\mathbb{P}_{X_0}$, we observe in the large time limit*
$$m_t := \mathbb{E}[\mathcal{N}_t] \sim \int_0^t \varphi_{1,2}(s)\mu_1(s)\,ds,$$
where μ_1 is the first coordinate of the periodic stationary measure associated with the Markov chain $(X_t)_{t\geq 0}$. In particular,
$$\lim_{t\to\infty} \frac{1}{t}\,\mathbb{E}[\mathcal{N}_t] = \frac{1}{T}\int_0^T \varphi_{1,2}(s)\mu_1(s)\,ds \tag{2.3}$$
2. Second moment. *Let us denote by $\mathcal{R}_\nu(t) := \mathrm{Var}(\mathcal{N}_t) - \mathbb{E}[\mathcal{N}_t]$ for the initial distribution of the Markov chain: $\mathbb{P}_{X_0} = \nu$. Then*
$$\mathcal{R}_{\mu(0)}(T) = 2\int_0^T \varphi_{1,2}(s)e_1^*\,P\,\Xi(s)\check{\mathrm{Id}}^1 C(s)\,ds, \tag{2.4}$$

where μ is the PSPM and

$$C(t) = \int_0^t \Xi(s)^{-1} P^{-1} B_s \mu(s)\, ds. \tag{2.5}$$

Moreover the following limit holds

$$\lim_{t\to\infty} \frac{1}{t}\,\mathcal{R}_\nu(t) = \frac{2}{T}\left\{ e_1^* \Big(\int_0^T \varphi_{1,2}(s) P\, \Xi(s)\, ds \Big) \Xi(T) \check{\mathrm{Id}}^1 \Big(\mathrm{Id} - \Xi(T)\check{\mathrm{Id}}^1 \Big)^{-1} C(T) \right\}$$
$$+ \frac{1}{T}\,\mathcal{R}_{\mu(0)}(T). \tag{2.6}$$

The mathematical quantity $\mathcal{R}_\nu(t)$ has been studied since its expression is actually more concise than the explicit expression of the variance. Moreover, it is well known that the variance of a Poisson distribution equals its average. Therefore $\mathcal{R}$ vanishes for this particular probability distribution which in fact plays an important role for counting processes in homogeneous environments.

Remark 2.2. *1. The limit* (2.6) *does not depend on the initial distribution of X_0. This property is related to the ergodic behaviour of the Markov chain developed in 1.2.*
2. If the fundamental solution of (2.1) *at time T is diagonalizable, that is $r_1 = 1 > |r_2| > \ldots > |r_d|$ where r_i are the Floquet multipliers of* (1.1)*, then* (2.6) *takes a simpler form due to the following expression:*

$$\Big(\Xi(T)\check{\mathrm{Id}}^1 (\mathrm{Id} - \Xi(T)\check{\mathrm{Id}}^1)^{-1} \Big)_{i,j} = \frac{r_i}{1-r_i} 1_{\{i=j\geq 2\}}, \quad 1 \leq i,j \leq d.$$

3. If the transition probabilities are constant functions such that Q_t defined in (0.1) *satisfies*

$$\varphi_{i,j} = \varphi 1_{\{i\neq j\}} - (d-1)\varphi 1_{\{i=j\}},$$

for some constant $\varphi > 0$, then Theorem 1.2 can be applied for any $T > 0$ and straightforward computations lead to:

$$\mathcal{R}_{\mu(0)}(t) = \frac{2}{d^4}(1 - e^{-d\varphi t} - d\varphi t)$$

Hence

$$\lim_{t\to\infty} \frac{1}{t}\,\mathcal{R}_{\mu(0)}(t) = -\frac{2\varphi}{d^3}.$$

Even in this simple homogeneous situation, $\mathcal{N}_t$ is not asymptotically Poisson distributed. Indeed the Poisson distribution would satisfy $\mathcal{R}(t) = 0$.

Proof. **Step 1.** *Averaged number of transitions.* Let us first decompose the averaged number of transitions as follows:

$$m_t = \sum_{k=1}^d m_t^k \quad \text{with } m_t^k = \mathbb{E}[\mathcal{N}_t 1_{\{X_t = s_k\}}].$$

We set $\mathcal{M}_t := (m_t^1, \dots, m_t^d)^*$. For $h > 0$ small enough, we get

$$\begin{aligned}
m_{t+h}^2 =& \mathbb{E}[\mathcal{N}_{t+h} 1_{\{X_{t+h}=s_2\}}] = \sum_{1\le i\le d} \mathbb{E}[(\mathcal{N}_t + (\mathcal{N}_{t+h} - \mathcal{N}_t))1_{\{X_t=s_i,\ X_{t+h}=s_2\}}] \\
=& \sum_{1\le i\le d} \mathbb{E}[\mathcal{N}_t 1_{\{X_t=s_i\}}]\mathbb{P}(X_{t+h} = s_2 | X_t = s_i) \\
&+ \sum_{1\le i\le d} \mathbb{E}[(\mathcal{N}_{t+h} - \mathcal{N}_t)1_{\{X_t=s_i,\ X_{t+h}=s_2\}}] \\
=& m^2(t)(1 - \varphi_{2,2}(t)h) + \sum_{i=1, i\neq 2}^{d} m^i(t)\varphi_{i,2}(t)h + \nu_1(t)\varphi_{1,2}(t)h + o(h)
\end{aligned}$$

where $\nu_i(t) = \mathbb{P}(X_t = s_i)$. By similar computations, we obtain the result for $h < 0$ close to the origin. Moreover for $k \neq 2$:

$$\begin{aligned}
m_{t+h}^k =& \mathbb{E}[\mathcal{N}_{t+h} 1_{\{X_{t+h}=s_k\}}] \\
=& m^k(t)(1 - \varphi_{k,k}(t)h) + \sum_{i=1, i\neq k}^{d} m^i(t)\varphi_{i,k}(t)h + o(h).
\end{aligned}$$

Finally we observe that $\mathcal{M}_t$ satisfies the ode:

$$\dot{\mathcal{M}}_t = Q_t \mathcal{M}_t + B_t \nu_t, \quad \mathcal{M}_0 = 0, \tag{2.7}$$

where $(B_t)_{i,j} = \varphi_{1,2}(t)\delta_{i,2}\delta_{j,1}$. Let $M(t)$ the fundamental solution of (2.1). Since Q_t satisfies (H), $M(T)$ is an irreducible positive and stochastic matrix. Indeed, let us just explain why $\mathbb{P}_{s_i}(X_T = s_j) > 0$ for any i and j: let us assume that this inequality does not hold. Then for h small enough, there exists a state s_l such that

$$\mathbb{P}_{s_i}(X_{T-h} = s_l) > 0, \tag{2.8}$$

and

$$\mathbb{P}(X_T = s_j | X_{T-h} = s_l) = \varphi_{l,j}(T - h)h + o(h) \tag{2.9}$$

if $l \neq j$, otherwise:

$$\mathbb{P}(X_T = s_j | X_{T-h} = s_j) = 1 - \varphi_{j,j}(T - h)h + o(h). \tag{2.10}$$

By (H), the combination of (2.8), (2.9) and (2.10) leads to the announced property $\mathbb{P}_{s_i}(X_T = s_j) > 0$, as a product of two positive quantities. Therefore the Perron-Frobenius theorem (see chapter 8 in [15]) applied to the matrix $M(T)$ implies

- the eigenvalues $r_1, r_2, \dots, r_s$, $s \le d$ of the matrix $M(T)$ have the associated multiplicity $n_1 = 1$, $\sum_{k=1}^{s} n_k = d$ and $r_1 = 1 > |r_2| \ge \dots |r_s|$.
- the eigenvector associated to the first eigenvalue corresponds to the periodic stationary probability measure $\mu(0)$.

We denote therefore $\mathcal{B} = (\xi_1^0, \dots, \xi_d^0)$ the basis of the canonical Jordan form of the matrix $M(T)$ and P the basis matrix of $\mathcal{B}$, $P^{-1}M(T)P$ being then the Jordan form. In particular $\xi_1^0 = \mu(0)$. We define $\xi_k(t) = M(t)\xi_k^0$, $1 \le k \le d$ and observe two different cases. First case: ξ_k^0 is an eigenvector of $M(T)$ associated to the eigenvalue r_j which implies that

$$\xi_k(t+T) = M(t+T)\xi_k^0 = M(t)M(T)\xi_k^0 = r_j M(t)\xi_k^0 = r_j\xi_k(t) \qquad (2.11)$$

and consequently ξ_k is a Floquet solution associated to the Floquet multiplier r_j.

Second case: ξ_k^0 is not an eigenvector of $M(T)$ and belongs to the Jordan block associated to the eigenvalue r_j then

$$\xi_k(t+T) = M(t)M(T)\xi_k^0 = r_j M(t)\xi_k^0 + M(t)\xi_{k-1}^0 = r_j\xi_k(t) + \xi_{k-1}(t). \quad (2.12)$$

Furthermore we denote by $\Xi(t)$ the matrix defined by (2.2): the coefficient $\Xi_{i,j}(t)$ represents the i-th coordinate of the solution $\xi_j(t)$ in the basis $\mathcal{B}$ for $1 \le i, j \le d$. Let us note that since ξ_1 is a probability measure, $(1,\dots,1)\xi_1 = 1$. Moreover combining (1.1) and (1.2) leads to the following property: $(1,\dots,1)P\,\Xi(t)$ is a constant function. If ξ_k^0 is an eigenvector of $M(T)$ associated to the eigenvalue r_j then $\xi_k(T) = r_j\xi_k(0)$ with $|r_j| < 1$. In particular, since $(1,\dots,1)\xi_k(t)$ is constant in the canonical basis $(1,\dots,1)\xi_k(t) = 0$. If ξ_k^0 is not an eigenvector but belongs to the Jordan block associated to the eigenvalue r_j then (2.12) leads to

$$(1,\dots,1)\xi_k(T) = r_j(1,\dots,1)\xi_k(T) + (1,\dots,1)\xi_{k-1}(T).$$

If $(1,\dots,1)\xi_{k-1}(T) = 0$ then the property $|r_j| < 1$ leads to $(1,\dots,1)\xi_k(T) = 0$. So step by step, we prove that

$$(1,\dots,1)P\,\Xi(t) = (1,0,\dots,0), \quad \forall t \ge 0. \qquad (2.13)$$

Let us now solve the homogeneous part of the equation (2.7): there exists a vector $C = (C_1,\dots,C_d)^*$ such that

$$\mathcal{M}_t = P\,\Xi(t)C.$$

By the method of parameter variation, we obtain the system:

$$P\,\Xi(t)\dot{C}(t) = B_t\nu(t) = (0, \varphi_{1,2}(t)\nu_1(t), 0, \dots, 0)^*. \qquad (2.14)$$

The initial condition $\mathcal{M}(0) = 0$ leads to $C(0) = 0$. By multiplying (2.14) on the left side by the vector $(1,\dots,1)$ we obtain $\dot{C}_1(t) = \varphi_{1,2}(t)\nu_1(t)$. Hence

$$C_1(t) = \int_0^t \varphi_{1,2}(s)\nu_1(s)ds. \qquad (2.15)$$

We obtain therefore an explicit solution of (2.7) and deduce that

$$\mathbb{E}[\mathcal{N}_t] = (1,\dots,1)\mathcal{M}_t = C_1(t) = \int_0^t \varphi_{1,2}(s)\nu_1(s)ds \sim \int_0^t \varphi_{1,2}(s)P\,\Xi_{1,1}(s)ds,$$

as t becomes large. The equivalence presented in the previous equation is due to the ergodic property of the periodically driven Markov chain (Theorem 1.2). More precisely, for any sufficiently small constant $\epsilon > 0$ (smaller than $|\mathrm{Re}(\lambda_2)|$) there exists a constant $C > 0$ such that:

$$\begin{aligned}\Big|\mathbb{E}[\mathcal{N}_t] - \int_0^t \varphi_{1,2}(s)P\,\Xi_{1,1}(s)ds\Big| &\le \int_0^t \varphi_{1,2}(s)|\nu_1(s) - P\,\Xi_{1,1}(s)|ds \\ &\le C\int_0^t \varphi_{1,2}(s)e^{(\mathrm{Re}(\lambda_2)+\epsilon)s}ds. \qquad (2.16)\end{aligned}$$

Since the function $\varphi_{1,2}$ is bounded, so is the difference $\mathbb{E}[\mathcal{N}_t]-\int_0^t \varphi_{1,2}(s)P\,\Xi_{1,1}(s)ds$.

Step 2. *Description of the function $C(t)$.* From now on, we assume that the initial probability measure of the Markov chain is $\mu(0)$, the initial value of the PSPM. Before developing the asymptotics of the variance of $\mathcal{N}_t$ in the large time limit, we need to specify the function $C(t)$, solution of (2.14) where ν_1 is replaced by μ_1. We know that $C(0)=0$. Let us define

$$\eta(t) := C(t+T) - C(T) \quad \text{for any} \quad t \geq 0.$$

We observe that, due to the periodic property of ξ_1 and $\varphi_{1,2}$, the function η is solution of the following equation

$$P\,\Xi(t+T)\dot{\eta}(t) = B_t\xi_1(t), \quad \eta(0)=0. \tag{2.17}$$

Introducing $\tilde{\eta}(t) = \Xi(T)\eta(t)$, we obtain

$$P\,\Xi(t+T)\dot{\eta}(t) = P\,\Xi(t)\Xi(T)\dot{\eta}(t) = P\,\Xi(t)\dot{\tilde{\eta}}(t) = B_t\xi_1(t), \quad \tilde{\eta}(0)=0.$$

By uniqueness of the solution of the previous equation (Cauchy-Lipschitz theorem), the equality $\tilde{\eta}(t) = C(t)$ holds. Since $\Xi(T)$ is invertible (the Floquet multipliers are not equal to 0):

$$\eta(t) = C(t+T) - C(T) = \Xi(T)^{-1}C(t), \quad t \geq 0.$$

Therefore, using the definition of $\eta(t)$ and an iteration procedure, we deduce

$$C(t+lT) = \Big(\sum_{i=0}^{l-1}\Xi(T)^{-i}\Big)C(T) + \Xi(T)^{-l}C(t), \quad l \geq 1. \tag{2.18}$$

Step 3. *Asymptotics of the variance.* We now describe the asymptotic behaviour of the second moment. Let us denote $V_t = (v_t^1,\dots,v_t^d)^*$ with $v_t^k = \mathbb{E}[\mathcal{N}_t^2 1_{\{X_t=s_k\}}]$. Using similar arguments as those presented in the beginning of Step 1, we obtain the following differential equation:

$$\dot{V}_t = Q_tV_t + B_t(2\mathcal{M}_t + \mu(t)), \quad V_0 = 0. \tag{2.19}$$

The procedure is similar as above, the variation of parameters leads to:

$$V_t = P\,\Xi(t)\kappa(t) \quad \text{with} \quad \kappa(t) = (\kappa_1(t),\dots,\kappa_d(t))^*.$$

The coefficient $\kappa(t)$ is solution to the equation:

$$P\,\Xi(t)\dot{\kappa}(t) = B_t(2\mathcal{M}_t + \mu(t)), \quad \kappa(0)=0.$$

Multiplying the previous equation on the left-hand side by $(1,\dots,1)$ implies:

$$\dot{\kappa}_1(t) = \varphi_{1,2}(t)(2m_t^1 + \mu(t)), \quad \kappa_1(0)=0.$$

The second moment of the number of transitions between the states s_1 and s_2 satisfies $\mathbb{E}_\mu[\mathcal{N}_t^2] = (1,\dots,1)V_t = \kappa_1(t)$, that is:

$$\mathbb{E}_\mu[\mathcal{N}_t^2] = \int_0^t \varphi_{1,2}(s)(2m_s^1 + \mu(s))\,ds.$$

Here $\mathbb{E}_\mu$ stands for the expectation of the Markov chain distribution with the initial probability distribution $\mu(0)$. Let us set the vector $e_1 = (1, 0 \dots, 0)^*$ and the matrix $\check{\mathrm{Id}}^1_{i,j} = 1_{\{i=j\geq 2\}}$. On the one hand we have

$$\mathbb{E}_\mu[\mathcal{N}_t^2] = \int_0^t \varphi_{1,2}(s)\Big(2e_1^*\mathcal{M}_s + \mu_1(s)\Big)ds = \int_0^t \varphi_{1,2}(s)\Big(2e_1^* P\,\Xi(s)C(s) + \mu_1(s)\Big)ds.$$

On the other hand,

$$\begin{aligned}\mathbb{E}_\mu[\mathcal{N}_t]^2 &= \int_0^t 2m_s' m_s\,ds = \int_0^t 2\varphi_{1,2}(s)\mu_1(s)C_1(s)ds \\ &= \int_0^t 2\varphi_{1,2}(s)e_1^* P\,\Xi(s)(C(s) - \check{\mathrm{Id}}^1 C(s))ds.\end{aligned}$$

Hence,

$$\mathcal{R}_{\mu(0)}(t) := \mathrm{Var}_\mu(\mathcal{N}_t) - \mathbb{E}_\mu[\mathcal{N}_t] = 2\int_0^t \varphi_{1,2}(s)e_1^* P\,\Xi(s)\check{\mathrm{Id}}^1 C(s)ds. \tag{2.20}$$

Let us now compute the limit of the following expression $\mathcal{R}_{\mu(0)}(t)/t$ as $t \to \infty$. We first observe that $\Xi(T)\check{\mathrm{Id}}^1\Xi(T)^{-1} = \check{\mathrm{Id}}^1$ since $\Xi(T)$ is a Jordan canonical form with a first eigenvalue which is simple. By (2.18), we obtain, for $l > 0$,

$$\begin{aligned}\Delta(t,l) &:= P\,\Xi(t+lT)\check{\mathrm{Id}}^1 C(t+lT) \\ &= P\,\Xi(t)P^{-1}(P\,\Xi(T)P^{-1})^l P\,\check{\mathrm{Id}}^1\Big[\Big(\sum_{i=0}^{l-1}\Xi(T)^{-i}\Big)C(T) + \Xi(T)^{-l}C(t)\Big] \\ &= P\,\Xi(t)\check{\mathrm{Id}}^1\sum_{i=1}^{l}\Xi(T)^i C(T) + P\,\Xi(t)\check{\mathrm{Id}}^1 C(t) \\ &= P\,\Xi(t)\Big(\sum_{i=1}^{l}(\Xi(T)\check{\mathrm{Id}}^1)^i\Big)C(T) + P\Xi(t)\check{\mathrm{Id}}^1 C(t).\end{aligned} \tag{2.21}$$

The Perron-Frobenius theorem implies that the spectral radius $\rho(\Xi(T)\check{\mathrm{Id}}^1) = |r_2| < 1$. Due to Householder's theorem (see, for instance, Section 7.2 in [20]), there exists an induced norm satisfying $\|\Xi(T)\check{\mathrm{Id}}^1\| < 1$. Hence

$$\|P\,\Xi(t+lT)\check{\mathrm{Id}}^1 C(t+lT)\| \leq \frac{\|P\,\Xi(t)\|}{1 - \|\Xi(T)\check{\mathrm{Id}}^1\|}\,\|C(T)\| + \|P\,\Xi(t)\check{\mathrm{Id}}^1 C(t)\|.$$

The previous upper-bound does not depend on l. Moreover $t \mapsto P\,\Xi(t)\check{\mathrm{Id}}^1 C(t)$ is a bounded function on $[0,T]$. Therefore $\|P\,\Xi(t+lT)\check{\mathrm{Id}}^1 C(t+lT)\|$ is bounded for any l and $t \in [0,T]$. We deduce that $t \mapsto \varphi_{1,2}(t)e_1^* P\,\Xi(t)\check{\mathrm{Id}}^1 C(t)$, the function appearing in the integral (2.20) is also bounded. The limit we need to compute is then given by

$$\lim_{t\to\infty}\frac{1}{t}\,\mathcal{R}_{\mu(0)}(t) = \lim_{n\to\infty}\frac{2}{nT}\,I_n \quad \text{with} \quad I_n = \int_0^{nT}\varphi_{1,2}(s)e_1^* P\,\Xi(s)\check{\mathrm{Id}}^1 C(s)\,ds, \tag{2.22}$$

where T is the period of Q_t. Let us introduce

$$\mathbb{I}_0 = \int_0^T \varphi_{1,2}(s) P\,\Xi(s)\,ds \quad \text{and} \quad \mathbb{I}_1 = \int_0^T \varphi_{1,2}(s) P\,\Xi(s)\check{\mathrm{Id}}^1 C(s)\,ds$$

By (2.21) and since $\varphi_{1,2}$ is a periodic function, the following splitting holds

$$\begin{aligned}
I_n &= \sum_{k=0}^{n-1} \int_0^T \varphi_{1,2}(s) e_1^* P\,\Xi(s+kT)\check{\mathrm{Id}}^1 C(s+kT)\,ds \\
&= \sum_{k=1}^{n-1} \int_0^T \varphi_{1,2}(s) e_1^* P\,\Xi(s)\,ds \Big(\sum_{i=1}^{k} (\Xi(T)\check{\mathrm{Id}}^1)^i \Big) C(T) \\
&+ n \int_0^T \varphi_{1,2}(s) e_1^* P\,\Xi(s)\check{\mathrm{Id}}^1 C(s)\,ds \\
&= \sum_{k=1}^{n-1} e_1^* \mathbb{I}_0 \Big(\Xi(T)\check{\mathrm{Id}}^1 - (\Xi(T)\check{\mathrm{Id}}^1)^{k+1} \Big) \Big(\mathrm{Id} - \Xi(T)\check{\mathrm{Id}}^1 \Big)^{-1} C(T) + n e_1^* \mathbb{I}_1 \\
&= (n-1) e_1^* \mathbb{I}_0 \Xi(T)\check{\mathrm{Id}}^1 \Big(\mathrm{Id} - \Xi(T)\check{\mathrm{Id}}^1 \Big)^{-1} C(T) + n e_1^* \mathbb{I}_1 \\
&- e_1^* \mathbb{I}_0 \sum_{k=1}^{n-1} (\Xi(T)\check{\mathrm{Id}}^1)^{k+1} \Big(\mathrm{Id} - \Xi(T)\check{\mathrm{Id}}^1 \Big)^{-1} C(T).
\end{aligned}$$

Using the existence of an induced norm satisfying $\|\Xi(T)\check{\mathrm{Id}}^1\| < 1$, we deduce that the last term in the previous equality is bounded with respect to the variable n. Consequently (2.22) implies

$$\lim_{t\to\infty} \frac{1}{t}\,\mathcal{R}_{\mu(0)}(t) = \frac{2}{T} \left\{ e_1^* \mathbb{I}_0 \Xi(T)\check{\mathrm{Id}}^1 \Big(\mathrm{Id} - \Xi(T)\check{\mathrm{Id}}^1 \Big)^{-1} C(T) + e_1^* \mathbb{I}_1 \right\}.$$

Step 4. *Generalization to any initial probability distribution.* To end the proof, we are going to develop the idea that the initial distribution of the Markov chain does not play any role. The first part of the statement (Step 1) implies directly that

$$\lim_{t\to\infty} \frac{1}{t}\,\mathbb{E}_\nu(\mathcal{N}_t) = \lim_{t\to\infty} \frac{1}{t}\,\mathbb{E}_{\mu(0)}(\mathcal{N}_t).$$

Let us now observe the variance case. Let $n \in \mathbb{N}^*$, we introduce

$$\begin{cases} \Delta_n(t) := |\mathrm{Var}_\nu(\mathcal{N}_{nT+t}) - \mathrm{Var}_{\nu(nT)}(\mathcal{N}_t)|, \\ \Gamma_n(t) := |\mathrm{Var}_{\nu(nT)}(\mathcal{N}_t) - \mathrm{Var}_{\mu(0)}(\mathcal{N}_t)|, \end{cases} \tag{2.23}$$

where $\nu(nT)$ is the distribution of X_{nT} with the initial condition $\mathbb{P}_{X_0} = \nu$. Obviously

$$\lim_{t\to\infty} \frac{\mathrm{Var}_\nu(\mathcal{N}_\mathrm{t})}{t} = \lim_{t\to\infty} \frac{\mathrm{Var}_{\mu(0)}(\mathcal{N}_t)}{t}, \tag{2.24}$$

if

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t}\,\Delta_n(t) = \lim_{n\to\infty} \lim_{t\to\infty} \frac{1}{t}\,\Gamma_n(t) = 0.$$

Using the Markov property, we get

$$\begin{aligned}
\mathrm{Var}_\nu(\mathcal{N}_{nT+t}) &= \mathrm{Var}_\nu(\mathcal{N}_{nT+t} - \mathcal{N}_{nT}) + \mathrm{Var}_\nu(\mathcal{N}_{nT}) + 2\mathrm{Cov}_\nu(\mathcal{N}_{nT+t} - \mathcal{N}_{nT}, \mathcal{N}_{nT}) \\
&= \mathrm{Var}_{\nu(nT)}(\mathcal{N}_t) + \mathrm{Var}_\nu(\mathcal{N}_{nT}) + 2\mathrm{Cov}_\nu(\mathcal{N}_{nT+t} - \mathcal{N}_{nT}, \mathcal{N}_{nT}).
\end{aligned} \tag{2.25}$$

Moreover, let us introduce:

$$K(n,\nu) := \max_{z\in\mathcal{S}} \Big| \mathbb{E}_\nu[\mathcal{N}_{nT}|X_{nT}=z] - \mathbb{E}_\nu[\mathcal{N}_{nT}] \Big|.$$

If we denote by $\mathbb{E}_x$ the conditional expectation under the event $\{X_0 = x\}$, we observe that

$$\begin{aligned}
\overline{\Delta}_n(t) &:= |\mathrm{Cov}_\nu(\mathcal{N}_{nT+t} - \mathcal{N}_{nT}, \mathcal{N}_{nT})| \\
&= \Big| \sum_{x\in\mathcal{S}} \mathbb{E}_x[\mathcal{N}_t] \Big(\mathbb{E}_\nu[\mathcal{N}_{nT}|X_{nT}=x] - \mathbb{E}_\nu[\mathcal{N}_{nT}] \Big) \nu_x(nT) \Big| \\
&= \Big| \sum_{x\in\mathcal{S}} \Big(\mathbb{E}_x[\mathcal{N}_t] - \mathbb{E}_{\nu(nT)}[\mathcal{N}_t] \Big) \Big(\mathbb{E}_\nu[\mathcal{N}_{nT}|X_{nT}=x] - \mathbb{E}_\nu[\mathcal{N}_{nT}] \Big) \nu_x(nT) \Big| \\
&\le K(n,\nu) \sum_{(x,y)\in\mathcal{S}^2} \Big| \mathbb{E}_x[\mathcal{N}_t] - \mathbb{E}_y[\mathcal{N}_t] \Big| \nu_x(nT)\nu_y(nT) \\
&\le 2K(n,\nu) \max_{x\in\mathcal{S}} \Big| \mathbb{E}_x[\mathcal{N}_t] - \mathbb{E}_{\mu(0)}[\mathcal{N}_t] \Big|.
\end{aligned}$$

By (2.3) the normalized averages appearing in the last upper-bound are equivalent in the large time scale, the following asymptotic result therefore holds

$$\lim_{t\to\infty} \frac{1}{t}\, \overline{\Delta}_n(t) = 0. \tag{2.26}$$

Consequently, combining (2.25) and (2.26) leads to $\lim_{t\to\infty} \frac{1}{t}\, \Delta_n(t) = 0$. Finally let us prove that $\lim_{n\to\infty} \lim_{t\to\infty} \Gamma_n(t)/t = 0$ in order to prove (2.24). Due to the Perron-Frobenius theorem, the PSPM satisfies $\mu_x(0) > 0$ for any $x \in \mathcal{S}$ and so, using the definition of Γ_n in (2.23), we obtain

$$\begin{aligned}
\Gamma_n(t) &= \sum_{x\in\mathcal{S}} \mathrm{Var}_x(\mathcal{N}_t) \Big| \frac{\nu_x(nT)}{\mu_x(0)} - 1 \Big| \mu_x(0) \\
&\le \frac{\|\nu(nT) - \mu(0)\|}{\min_{x\in\mathcal{S}} \mu_x(0)} \mathrm{Var}_{\mu(0)}(\mathcal{N}_t) \\
&= \frac{\|\nu(nT) - \mu(nT)\|}{\min_{x\in\mathcal{S}} \mu_x(0)} \mathrm{Var}_{\mu(0)}(\mathcal{N}_t).
\end{aligned}$$

Combining Step 3 in order to describe the asymptotic behaviour of $\mathrm{Var}_{\mu(0)}(\mathcal{N}_t)$ and Theorem 1.2 permits to imply $\lim_{n\to\infty} \lim_{t\to\infty} \Gamma_n(t)/t = 0$ and consequently (2.24). □

The particular 2-dimensional case

The aim of this section is to express the statement of Theorem 2.1 in the situation $\mathcal{S} = \{s_1, s_2\}$. The results obtained in this quite simple situation are not trivial and can be clarified since the explicit expression of the *periodic stationary probability measure* has been developed in Proposition 1.3.

Corollary 2.3. *1. The number of transitions between state s_1 and state s_2, denoted by $\mathcal{N}_t$, satisfies*

$$\lim_{t\to\infty} \frac{1}{t}\, \mathbb{E}[\mathcal{N}_t] = \frac{1}{T} \int_0^T \varphi_{1,2}(s)\mu_1(s)\, ds,$$

where $(\mu(t))_{t\geq 0}$ is the PSPM (1.6). *This result does not depend on the initial distribution of the Markov chain (X_t).*
2. Moreover the following large time limit for the variance holds:

$$\lim_{t\to\infty} \frac{1}{t}\Big(\mathrm{Var}(\mathcal{N}_t) - \mathbb{E}[\mathcal{N}_t]\Big) = -\frac{2}{T}\frac{\mathbb{J}_1(T)\mathbb{J}_2(1)}{e^{-\lambda_2 T}-1} - \frac{2}{T}\mathbb{J}_2(\mathbb{J}_1).$$

where $\lambda_2 = -\frac{1}{T}\int_0^T \varphi_{1,2}(t) + \varphi_{2,1}(t)\,dt$ is the second Floquet exponent,

$$\mathbb{J}_1(t) = \int_0^t \frac{\varphi_{1,2}(s)\mu_1^2(s)}{\zeta_1(s)}\,ds, \quad \mathbb{J}_2(f) := \int_0^T \varphi_{1,2}(s)\zeta_1(s)f(s)\,ds,$$

$$\zeta_1(t) = -\exp\left(-\int_0^t (\varphi_{1,2}+\varphi_{2,1})(s)ds\right), \quad t \geq 0. \tag{2.27}$$

Proof. It suffices to apply Theorem 2.1. Considering the arguments used in Proposition 1.3, we know explicitly the fundamental solution of (2.1). In particular the Jordan canonical form $\Xi(T)$ (defined in (2.2)) is given by

$$\Xi(T) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda_2 T} \end{pmatrix},$$

where λ_2 is defined in (1.9). The Floquet solution associated to the multiplier 1 is the PSPM (1.6) and the Floquet solution associated to the multiplier $e^{\lambda_2 T}$ is $\zeta(t)^* = (\zeta_1(t), -\zeta_1(t))$ with $\zeta_1(t)$ defined in (2.27). We deduce that the basis matrix associated with the Jordan matrix is:

$$P = \begin{pmatrix} \mu_1(0) & -1 \\ 1-\mu_1(0) & 1 \end{pmatrix}, \quad \text{with} \quad \mu_1(0) = \frac{I(\varphi_{2,1})}{I(\varphi_{1,2}+\varphi_{2,1})}.$$

The function I has been defined in (1.7). Consequently

$$P\,\Xi(t) = \begin{pmatrix} \mu_1(t) & \zeta_1(t) \\ 1-\mu_1(t) & -\zeta_1(t) \end{pmatrix}, \quad t \geq 0.$$

The $\mathbb{R}^2$-valued function C defined by (2.5) is equal to

$$C_1(t) = \int_0^t \varphi_{1,2}(s)\mu_1(s)\,ds, \quad C_2(t) = -\int_0^t \frac{\mu_1(s)^2\varphi_{1,2}(s)}{\zeta_1(s)}\,ds.$$

All these explicit expressions and simple computations combined with (2.4) and (2.6) imply the announced statement. □

2.2 Positivity of the limit for the normalized variance

Theorem 2.1 and Corollary 2.4 ensure that the limit, in the large time scale, of the normalized variance $\mathrm{Var}(\mathcal{N}_t)/t$ exists. The expression of the limit is quite general and can be computed explicitly in any particular situation. One important property concerning this limit is the positivity. This step is crucial as a preliminary result for the proof of a Central Limit Theorem for the statistics $\mathcal{N}_t$.

Proposition 2.4. *Under the hypothesis* (H)*, the long time limit of the normalized variance is positive:*

$$\lim_{t\to\infty} \frac{\mathrm{Var}(\mathcal{N}_t)}{t} > 0 \tag{2.28}$$

Proof. Let us decompose $\mathcal{N}_{kT}$ into

$$\mathcal{N}_{kT} = \sum_{j=1}^{k} \Delta\mathcal{N}_j \quad \text{with} \quad \Delta\mathcal{N}_j = \mathcal{N}_{jT} - \mathcal{N}_{(j-1)T}.$$

Using the conditioning with respect to the position of the Markov chain at times $0, T, \ldots, kT$, we define $\mathbb{X}_k := (X_0, X_T, \ldots, X_{kT})$ and obtain

$$\mathrm{Var}(\mathcal{N}_{kT}) = \mathrm{Var}\Big(\mathbb{E}[\mathcal{N}_{kT}|\mathbb{X}_k]\Big) + \mathbb{E}\Big[\mathrm{Var}(\mathcal{N}_{kT}|\mathbb{X}_k)\Big] \geq \mathbb{E}\Big[\mathrm{Var}(\mathcal{N}_{kT}|\mathbb{X}_k)\Big]. \tag{2.29}$$

We just recall that the conditional variance is given by:

$$\begin{aligned}
\mathrm{Var}(\mathcal{N}_{kT}|\mathbb{X}_k) &= \mathbb{E}\Big[(\mathcal{N}_{kT} - \mathbb{E}[\mathcal{N}_{kT}|\mathbb{X}_k])^2\Big|\mathbb{X}_k\Big] \\
&= \mathbb{E}\Big[\Big(\sum_{j=1}^{k} \Delta\mathcal{N}_j - \mathbb{E}[\Delta\mathcal{N}_j|\mathbb{X}_k]\Big)^2\Big|\mathbb{X}_k\Big].
\end{aligned}$$

Developing the square implies:

$$\begin{aligned}
\mathrm{Var}(\mathcal{N}_{kT}|\mathbb{X}_k) &= \mathbb{E}\Big[\sum_{j=1}^{k}\Big(\Delta\mathcal{N}_j - \mathbb{E}[\Delta\mathcal{N}_j|\mathbb{X}_k]\Big)^2\Big|\mathbb{X}_k\Big] \\
&\quad + 2\sum_{1\leq j<l\leq k} \mathbb{E}\Big[\Big(\Delta\mathcal{N}_j - \mathbb{E}[\Delta\mathcal{N}_j|\mathbb{X}_k]\Big)\Big(\Delta\mathcal{N}_l - \mathbb{E}[\Delta\mathcal{N}_l|\mathbb{X}_k]\Big)\Big|\mathbb{X}_k\Big].
\end{aligned}$$

Given $\mathbb{X}_k$, the random variables $\Delta\mathcal{N}_j - \mathbb{E}[\Delta\mathcal{N}_j|\mathbb{X}_k]$ and $\Delta\mathcal{N}_l - \mathbb{E}[\Delta\mathcal{N}_l|\mathbb{X}_k]$ are independent and centred (for $1 \leq j < l \leq k$). Consequently the double sum in the previous equality vanishes. The Markov property leads to

$$\mathrm{Var}(\mathcal{N}_{kT}|\mathbb{X}_k) = \sum_{j=1}^{k} \mathrm{Var}(\Delta\mathcal{N}_j|\mathbb{X}_k) = \sum_{j=1}^{k} \mathrm{Var}(\Delta\mathcal{N}_j|X_{(j-1)T}, X_{jT}).$$

Let us define the function $\psi : \mathcal{S}\times\mathcal{S} \to \mathbb{R}_+$ by

$$\psi(a,b) = \mathrm{Var}(\Delta\mathcal{N}_j|X_{(j-1)T} = a, X_{jT} = b)$$

which does not depend on j since the transition probabilities are T-periodic. Since the state space is finite, the minimum of the function ψ is reached. Moreover the random variable $\Delta\mathcal{N}_1$ knowing both X_0 and X_T is not constant a.s. due to the hypothesis (H), so that the following minimum is positive:

$$V^* = \min_{(a,b)\in\mathcal{S}^2} \psi(a,b) > 0.$$

Therefore the following lower bound holds

$$\mathrm{Var}(\mathcal{N}_{kT}|\mathbb{X}_k) \geq kV^* \quad \text{and so} \quad \mathrm{Var}(\mathcal{N}_{kT}) \geq kV^*,$$

just by using (2.29). Dividing by kT, we obtain

$$\lim_{k\to\infty} \frac{\mathrm{Var}(\mathcal{N}_{kT})}{kT} > 0. \tag{2.30}$$

The statement of Theorem 2.1 points out that the limit considered in (2.28) exists and since the limit of a subsequence (2.30) is positive, we deduce the positivity of (2.28). □

2.3 Mixing properties of the time periodic Markov chain

We have already partially described, in the previous results, the behaviour of the Markov chain in the long time limit. The distribution of the Markov chain converges exponentially fast toward the unique periodic stationary probability measure, the normalized (divided by the time variable) averaged number of transitions between two given states converges, so does the normalized variance. All these results concern the one marginal distribution of the Markov chain (X_t) or the one marginal distribution of the counting process $(\mathcal{N}_t)$. In order to complete this study and to better understand the long time behaviour of X_t, we are going to prove that the Markov chain $(X_t)_{t\geq 0}$ is weakly correlated, that is, X_t and X_{t+h} are weakly dependent when h is large enough. This property can be measured with a particular tool associated to the *strong mixing* concept. This property is quite evident for homogeneous Markov chains, we prove here that it is also satisfied for periodic inhomogeneous Markov chains.

Let us first introduce the σ-algebra

$$\mathcal{F}_{i,i+j} = \sigma(\Delta\mathcal{N}_k : i \leq k \leq i+j)$$

where $\Delta\mathcal{N}_k := \mathcal{N}_{kT} - \mathcal{N}_{(k-1)T}$, and secondly, the mixing coefficients $\alpha_n(k)$ defined, for $k \leq n-1$, by

$$\alpha_n(k) = \sup\Big\{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| : \ A \in \mathcal{F}_{1,m}, B \in \mathcal{F}_{m+k,n}, \ 1 \leq m \leq n-k\Big\}.$$

We set $\alpha_n(k) = 0$ for $k \geq n$. These mixing coefficients permit to measure the dependence between random variables belonging to the same sequence. For periodic forced Markov chains, we prove that the dependence of X_t with respect to the initial condition rapidly decreases as time elapses. It is a consequence of the following result.

Proposition 2.5. *The family of random variables $(\Delta\mathcal{N}_k)_{k\in\mathbb{N}}$ is a strongly mixing sequence, that is*

$$\alpha(k) := \sup_{n\geq 1} \alpha_n(k) = O(b^{-k}) \quad \textit{for some } b > 1.$$

Proof. Let $A \in \mathcal{F}_{1,m}$ and $B \in \mathcal{F}_{m+k,n}$ then there exist two measurable bounded and non-negative functions ψ_A and ψ_B such that

$$1_A = \psi_A(\Delta\mathcal{N}_1, \dots, \Delta\mathcal{N}_m) \quad \text{and} \quad 1_B = \psi_B(\Delta\mathcal{N}_{m+k}, \dots, \Delta\mathcal{N}_n).$$

Then, due to the Markov property, we obtain

$$\begin{aligned}\mathbb{P}(A\cap B) &= \mathbb{E}[\psi_A(\Delta\mathcal{N}_1, \dots, \Delta\mathcal{N}_m)\psi_B(\Delta\mathcal{N}_{m+k}, \dots, \Delta\mathcal{N}_n)] \\ &= \mathbb{E}\Big[\psi_A(\Delta\mathcal{N}_1, \dots, \Delta\mathcal{N}_m)\psi_C(X_{(m+k-1)T})\Big],\end{aligned}$$

where ψ_C is a bounded non negative measurable function defined by

$$\begin{aligned}\psi_C(x) &= \mathbb{E}[\psi_B(\Delta\mathcal{N}_{m+k}, \ldots, \Delta\mathcal{N}_n)|X_{(m+k-1)T} = x] \\ &= \mathbb{E}_x[\psi_B(\Delta\mathcal{N}_1, \ldots, \Delta\mathcal{N}_{n+1-m-k})].\end{aligned}$$

We deduce that

$$\begin{aligned}\mathbb{P}(A \cap B) &= \mathbb{E}[\psi_A(\Delta\mathcal{N}_1, \ldots, \Delta\mathcal{N}_m)\psi_C(X_{(m+k-1)T})] \\ &= \sum_{i=1}^{d} \mathbb{E}[\psi_A(\Delta\mathcal{N}_1, \ldots, \Delta\mathcal{N}_m)1_{\{X_{(m+k-1)T}=s_i\}}]\psi_C(s_i) \\ &= \sum_{i,j=1}^{d} \mathbb{E}[\psi_A(\Delta\mathcal{N}_1, \ldots, \Delta\mathcal{N}_m)1_{\{X_{mT}=s_j\}}1_{\{X_{(m+k-1)T}=s_i\}}]\psi_C(s_i) \\ &= \sum_{i,j=1}^{d} \mathbb{E}[\psi_A(\Delta\mathcal{N}_1, \ldots, \Delta\mathcal{N}_m)1_{\{X_{mT}=s_j\}}]\mathbb{P}_j(X_{(k-1)T} = s_i)\psi_C(s_i).\end{aligned}$$

By similar computations, we obtain:

$$\begin{cases} \mathbb{P}(A) = \sum_{j=1}^{d} \mathbb{E}[\psi_A(\Delta\mathcal{N}_1, \ldots, \Delta\mathcal{N}_m)1_{\{X_{mT}=s_j\}}], \\ \mathbb{P}(B) = \sum_{i=1}^{d} \mathbb{P}(X_{(m+k-1)T} = s_i)\psi_C(s_i). \end{cases}$$

Finally $\Delta := |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|$ is equal to

$$\begin{aligned}\Delta = \Big| &\sum_{i,j=1}^{d} \mathbb{E}[\psi_A(\Delta\mathcal{N}_1, \ldots, \Delta\mathcal{N}_m)1_{\{X_{mT}=s_j\}}] \\ &\times \psi_C(s_i)\Big(\mathbb{P}(X_{(m+k-1)T} = s_i) - \mathbb{P}_{s_j}(X_{(k-1)T} = s_i)\Big)\Big| \\ &\le \max_{1\le i,j,l\le d} |\mathbb{P}_{s_l}(X_{(k-1)T} = s_i) - \mathbb{P}_{s_j}(X_{(k-1)T} = s_i)| \\ &\le 2 \max_{1\le i,l\le d} |\mathbb{P}_{s_j}(X_{(k-1)T} = s_i) - \mu_{s_i}(0)|\end{aligned}$$

where $\mu(t)$ is the *periodic stationary probability measure* associated with the chain (X_t). Due to the ergodic property (Theorem 1.2), for any initial probability measure ν we have

$$\lim_{t\to\infty} \frac{1}{t} \log \sup_{1\le i\le d} |\mathbb{P}(X_t = s_i) - \mu_{s_i}(t)| \le \mathrm{Re}(\lambda_2) < 0,$$

where λ_2 is the second Floquet exponent associated with the distribution of the periodically driven Markov chain. □

2.4 A Functional Central Limit Theorem for the averaged number of transitions

In this section, we aim to point out the main result of this study. We have already given some description of the long time asymptotics of the number of transitions $\mathcal{N}_t$, by computing the two first moments. Moreover the mixing property developed in Proposition 2.5 permits to ensure that the periodic forced Markov chain would behave in a quite similar way as a time-homogeneous Markov chain. In fact, we have to be careful since the law of X_t always depends on the initial condition (see Theorem 1.2).

Theorem 2.6. Functional Central Limit Theorem. *The stochastic process*

$$Z_n(t) := \frac{\mathcal{N}_{ntT} - \mathbb{E}[\mathcal{N}_{ntT}]}{\sqrt{\mathrm{Var}(\mathcal{N}_{ntT})}}, \quad t \in [0,1], \tag{2.31}$$

converges in distribution to the standard Brownian motion $(W_t,\ 0 \le t \le 1)$ *as* $n \to \infty$.

Let us just present the following preliminary result which corresponds to an important argument in the proof of Theorem 2.6. We first recall that $\varphi_{i,j}(t) \ge 0$ for all $1 \le i,j \le d$ and $t \in [0,T]$ and define:

$$M = \max_{t\in[0,T], i\neq j} \varphi_{i,j}(t) > 0. \tag{2.32}$$

Lemma 2.7. *Let us define* $\Delta\mathcal{N}_n := \mathcal{N}_{nT} - \mathcal{N}_{(n-1)T}$ *for* $n \ge 1$ *and introduce* $(\mathcal{P}_t)$ *a Poisson process of parameter* $(d-1)M$. *Then* $\mathcal{P}_T$ stochastically dominates *the random variable* $\Delta\mathcal{N}_n$*:*

$$\Delta\mathcal{N}_n \preceq \mathcal{P}_T \quad i.e. \quad \mathbb{P}(\mathcal{P}_T \ge r) \ge \mathbb{P}(\Delta\mathcal{N}_n \ge r), \quad \forall r \in \mathbb{N}.$$

Proof of Lemma 2.7. Let us prove this statement for the particular case $n = 1$ (similar arguments permit to deal with the general case). We set $\mathcal{T}_0 = 0$ and introduce $(\mathcal{T}_k,\ k \ge 1)$ the successive transition times of the periodically driven Markov chain (X_t) and $\Delta\mathcal{T}_k := \mathcal{T}_k - \mathcal{T}_{k-1}$. Of course, the number of transitions between the states 1 and 2 before time T is smaller than the number of transitions corresponding to the whole state space. That is why, for $r \in \mathbb{N}$, we get

$$\mathbb{P}(\Delta\mathcal{N}_1 \ge r) \le \mathbb{P}(\mathcal{T}_r \le T) \le \mathbb{P}(\Delta\mathcal{T}_1 + \ldots + \Delta\mathcal{T}_r \le T).$$

Moreover we define for $k \ge 1$:

$$\mathcal{U}_k(t,x) := \int_{\mathcal{T}_{k-1}}^{\mathcal{T}_{k-1}+t} \varphi_{x,x}(u)\,du, \quad t \ge 0, \quad x \in \mathcal{S}.$$

Combining (2.32) and (2.13) leads to $\mathcal{U}_k(t,x) \le (d-1)Mt$ for all x and t and in particular: $\mathcal{U}_k(\Delta\mathcal{T}_k, X_{\mathcal{T}_{k-1}}) \le (d-1)M\Delta\mathcal{T}_k$ for all $k \ge 1$. Furthermore straightforward computations permit to point out that

$$\mathcal{U}_1(\Delta\mathcal{T}_1, X_0),\ \mathcal{U}_2(\Delta\mathcal{T}_2, X_{\mathcal{T}_1}), \ldots, \mathcal{U}_k(\Delta\mathcal{T}_k, X_{\mathcal{T}_{k-1}}), \ldots$$

is a sequence of independent exponentially distributed random variables with mean 1. Finally we obtain

$$\begin{aligned}\mathbb{P}(\Delta\mathcal{N}_1 \ge r) &\le \mathbb{P}(\Delta\mathcal{T}_1 + \ldots + \Delta\mathcal{T}_r \le T)\\ &\le \mathbb{P}\Big(\frac{\mathcal{U}_1(\Delta\mathcal{T}_1, X_0)}{(d-1)M} + \frac{\mathcal{U}_2(\Delta\mathcal{T}_2, X_{\mathcal{T}_1})}{(d-1)M} + \ldots + \frac{\mathcal{U}_r(\Delta\mathcal{T}_r, X_{\mathcal{T}_{r-1}})}{(d-1)M} \le T\Big)\\ &= \mathbb{P}(\mathcal{P}_T \ge r), \quad r \in \mathbb{N}.\end{aligned}$$

□

Proof of Theorem 2.6. **Step 1.** The arguments developed in the first step of the proof are based on the application of Corollary 2 in [11]. Let us introduce $(Y_n)_{n\ge 1}$ defined by

$$Y_n = \Delta\mathcal{N}_n - \mathbb{E}[\Delta\mathcal{N}_n], \quad \Delta\mathcal{N}_n := \mathcal{N}_{nT} - \mathcal{N}_{(n-1)T}.$$

We just recall this result: the process $S_n := \sum_{k=1}^n Y_k$ satisfies the Central Limit Theorem (2.31) as soon as the following conditions are satisfied:

1. $\mathbb{E}[Y_n] = 0$ and $\mathbb{E}[Y_n^2] < \infty$ for any $n \ge 1$.
2. The sequence of normalized variances converges as $n \to \infty$:

$$\lim_{n\to\infty} \frac{\mathbb{E}[S_n^2]}{n} = \sigma^2 > 0 \quad \text{for some } \sigma > 0.$$

   Moreover

$$\sup\left\{\frac{1}{n}\ \mathbb{E}[(S_{m+n} - S_m)^2] :\ (n, m) \in \mathbb{N}^2\right\} < \infty. \tag{2.33}$$

3. There exists $\beta > 2$ (we set $\gamma = 2/\beta$) such that

$$\|Y_n\|_\beta = o\Big(n^{(1-\gamma)/2}/(\log n)^{1-\gamma/2}\Big) \quad \text{and} \quad \alpha(k) = O(b^{-k}), \tag{2.34}$$

   for some $b > 1$ and $\|Y_n\|_\beta = \mathbb{E}^{1/\beta}[|Y_n|^\beta]$.

Under these three conditions, $W_n(t) := S_{\lfloor nt\rfloor}/(\sigma\sqrt{n})$ converges in distribution towards a standard Brownian motion W. Let us now point out that these conditions are satisfied for the periodic driven Markov chain. The first condition is trivial. The second condition is directly related to the convergence pointed out in Proposition 2.4. Let us now prove (2.33): Lemma 2.7 ensures the stochastic dominance $\Delta\mathcal{N}_k \preceq \mathcal{P}_T$ where $(\mathcal{P}_t)$ is a Poisson process of parameter $(d-1)M$. Since all moments of a Poisson process are finite so are the moments of $\Delta\mathcal{N}_k$. We deduce immediately that $\|Y_n\|_\beta$ is a bounded sequence (the first part of (2.34) is therefore satisfied). Furthermore

$$\mathbb{E}\left[\frac{(S_{m+n} - S_n)^2}{n}\right] \le \mathbb{E}\left[\frac{\mathcal{N}_{nT}^2}{n}\right] \le (d-1)MT < \infty, \quad \forall (m, n) \in \mathbb{N}^2.$$

Finally let us note that the second part of (2.34) is an immediate consequence of Proposition 2.5.

Step 2. In the first step, the convergence in distribution of W_n towards W was emphasized. Now let us deduce the convergence of Z_n towards W. The following splitting holds

$$\begin{aligned}\mathcal{N}_{ntT} - \mathbb{E}[\mathcal{N}_{ntT}] = \mathcal{N}_{\lfloor nt\rfloor T} - \mathbb{E}[\mathcal{N}_{\lfloor nt\rfloor T}] + (\mathcal{N}_{ntT} - \mathcal{N}_{\lfloor nt\rfloor T})\\ - \mathbb{E}[\mathcal{N}_{ntT} - \mathcal{N}_{\lfloor nt\rfloor T}].\end{aligned} \tag{2.35}$$

Let us define the function $U_n : [0,1] \to \mathbb{N}$ by $U_n(t) = \mathcal{N}_{ntT} - \mathcal{N}_{\lfloor nt\rfloor T}$. This function vanishes at any time instant $t = k/n$ with $k \in \{0, 1, \dots n\}$. Moreover U_n is a non decreasing function on each interval $[(k-1)/n, k/n[$. Hence

$$\sup_{t\in[0,1]} U_n(t) = \max_{1\le k\le n}\left(\lim_{t\to k/n,\ t<k/n} U_n(t)\right) \le \max_{1\le k\le n} \Delta\mathcal{N}_k$$

where $\Delta\mathcal{N}_k$ is the total number of transitions observed during the time interval $[(k-1)T, kT[,\ k \in \{1,\dots n\}$. This number is stochastically dominated by a Poisson distributed random variable of parameter $\lambda = (d-1)MT$ where M is defined by (2.32). So we prove that $U_n/\sqrt{n}$ converges in probability to the zero function. Indeed for any $\varepsilon > 0$, we set $\delta_n = \varepsilon\sqrt{n}$ and obtain

$$\mathbb{P}\Big(\max_{1\le k\le n} \Delta\mathcal{N}_k \ge \delta_n\Big) \le 1 - \Big(\min_{a\in\mathcal{S}} \mathbb{P}_a(\Delta\mathcal{N}_1 \le \delta_n)\Big)^n$$
$$\le 1 - \Big(1 - \sum_{l\ge\lfloor\delta_n\rfloor+1} \frac{\lambda^l e^{-\lambda}}{l!}\Big)^n \le 1 - \Big(1 - \frac{\lambda^{\lfloor\delta_n\rfloor+1}}{(\lfloor\delta_n\rfloor+1)!}\Big)^n.$$

As n goes to ∞ the Stirling formula permits to prove that

$$\lim_{n\to\infty} \Big(1 - \frac{\lambda^{\lfloor\delta_n\rfloor+1}}{(\lfloor\delta_n\rfloor+1)!}\Big)^n = 1.$$

Consequently

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{U_n(t)}{\sqrt{n}} \ge \varepsilon\right) = 0.$$

Combining (2.35) with the following convergences as $n \to \infty$:

$$\frac{\mathcal{N}_{\lfloor nt\rfloor T} - \mathbb{E}[\mathcal{N}_{\lfloor nt\rfloor T}]}{\sigma\sqrt{n}} \xrightarrow{(d)} W_t, \quad \frac{\sigma\sqrt{n}}{\sqrt{\mathrm{Var}(\mathcal{N}_{ntT})}} \longrightarrow 1,$$
$$\frac{(\mathcal{N}_{ntT} - \mathcal{N}_{\lfloor nt\rfloor T})}{\sigma\sqrt{n}} \xrightarrow{\mathbb{P}} 0, \quad \frac{\mathbb{E}[\mathcal{N}_{ntT} - \mathcal{N}_{\lfloor nt\rfloor T}]}{\sqrt{\mathrm{Var}(\mathcal{N}_{ntT})}} \longrightarrow 0,$$

leads to (2.31). $\square$

3 Two examples in the stochastic resonance framework

We seek to describe the phenomenon of stochastic resonance. Let us introduce a continuous-time Markov chain X_t oscillating between two states $\{s_1, s_2\}$ according to a T-periodic infinitesimal generator Q_t. Then by varying the period, we observe that the behaviour of the chain changes and adopts *more or less* periodic paths. The aim in each example is to find the optimal period such that the behaviour of the paths looks like the most periodic as possible. That is why we shall introduce a criterion which measures the periodicity of any random path. We propose to use a criterion associated with the mean number of transition on a period (it should be also possible to propose a measure based on the minimal variance but we do not adopt this point of view in this study). The interesting tunings correspond to situations where this averaged number is close to the value 1.

3.1 An infinitesimal generator constant on each half period

In this first example, we consider T-periodic rates given by

$$\varphi_{1,2}(t) = \varphi_0 1_{\{0\le t<T/2\}} + \varphi_1 1_{\{T/2\le t<T\}} = \varphi_0 + \varphi_1 - \varphi_{2,1}(t). \tag{3.1}$$

where $\varphi_0 = p\, e^{-\frac{V}{\epsilon}}$ et $\varphi_1 = q\, e^{-\frac{v}{\epsilon}}$ with $p, q > 0$, ϵ, $0 < v < V$. This Markov model is often used in the stochastic resonance framework (see for instance [18]). Here we can compute explicitly the invariant measure (see also [18] Proposition 4.1.2 p.34)

Lemma 3.1. *The periodic stationary probability measure PSPM is given by:*

$$\mu_1(t) = \frac{e^{-(\varphi_0+\varphi_1)t}}{1+e^{-(\varphi_0+\varphi_1)T/2}} \frac{\varphi_0 - \varphi_1}{\varphi_0+\varphi_1} + \frac{\varphi_1}{\varphi_0+\varphi_1} \tag{3.2}$$

and $\mu_1(t)+\mu_2(t) = 1$, $\mu_1(t+T/2) = \mu_2(t)$, $\mu_2(t+T/2) = \mu_1(t)$. *Here* μ_1 *(resp.* μ_2*) stands for* μ_{s_1} *(resp.* μ_{s_2}*).*

Proof. Using the description of the PSPM in Proposition 1.3 we obtain

$$\begin{aligned}\mu_1(t) &= \mu_1(0)e^{-(\varphi_0+\varphi_1)t} + \frac{\varphi_1}{\varphi_0+\varphi_1}\Big(1-e^{-(\varphi_0+\varphi_1)t}\Big) \\ &= \left(\mu_1(0) - \frac{\varphi_1}{\varphi_0+\varphi_1}\right) e^{-(\varphi_0+\varphi_1)t} + \frac{\varphi_1}{\varphi_0+\varphi_1}, \quad 0 \le t < T/2. \end{aligned} \tag{3.3}$$

Furthermore, by symmetry arguments, the dynamics of the periodic invariant measure satisfies: $\mu_1(t+T/2) = \mu_2(t)$ for all $t \ge 0$. We deduce in particular that $\mu_1(T/2) = \mu_2(0) = 1 - \mu_1(0)$. Thus

$$\mu_1(0) = \frac{\varphi_0 + \varphi_1\, e^{-(\varphi_0+\varphi_1)T/2}}{(\varphi_0+\varphi_1)(1+e^{-(\varphi_0+\varphi_1)T/2})}$$

The equation (3.3) then permits to conclude. □

An immediate consequence of Corollary 2.3 leads to the explicit computation of the mean number of transitions (the details of the proof are left to the reader).

Proposition 3.2. *The mean number of transitions from state* s_1 *to state* s_2 *of the periodically driven Markov chain satisfies*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\mathcal{N}_{nT}] = \frac{\varphi_0\varphi_1 T}{\varphi_0+\varphi_1} + \left(\frac{\varphi_0-\varphi_1}{\varphi_0+\varphi_1}\right)^2 \tanh\Big((\varphi_0+\varphi_1)T/4\Big). \tag{3.4}$$

This expression represents the asymptotic averaged number of one-sided transitions during one period.

We are interested in the phenomenon of stochastic resonance associated to continuous-time process $(X_t,\, t \ge 0)$. This process essentially depends on two parameters: a parameter ϵ describing the intensity of the transition rates between both states $\{s_1, s_2\}$ (some small ϵ corresponds to a *frozen* situation: the Markov chain remains in the same state for a long while) and a second parameter T, the period of the process dynamics. By considering the normalized process $Y_t = X_{tT}$, especially its paths on a fixed interval $[0, S]$, we observe the following phenomenon (for fixed ϵ): if T is small then there are very few transitions of Y: the process tends to remain in its original state. If T is large, Y behaves in a chaotic way: lots of transitions are observed. For some intermediate values of T, the random paths of Y are close to deterministic periodic functions (one transition in each direction per period). Let us note that this phenomenon can also be observed by freezing the period length T and varying the intensity ϵ of the rates.

The aim is therefore to point out the best relationship (tuning) between ϵ and T which makes the process Y the most periodic as possible. If the process is close to a periodic function then the mean number of transition from state s_1 to sate s_2 is close to 1 per period. By Proposition 3.2, it is then sufficient to find the best relation between ϵ and T such that

$$\mathbb{E}_\mu[\mathcal{N}_T] = 1. \tag{3.5}$$

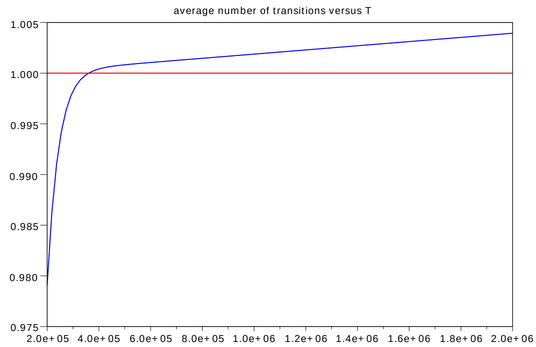

Figure 1: Average number of transitions. We set $\epsilon = 0.1$, $V = 2$, $v = 1$, $p = q = 1$ and let T vary. We compute numerically the average number of transitions per period. We can clearly observe that there is one and only one period corresponding to the condition (3.5).

Proposition 3.3. *Let T^ϵ_{opt} be the period which provides an average number of transitions per period equal to* 1*. The following asymptotic behaviour holds, as ϵ tends to* 0*,*

$$T^\varepsilon_{\text{opt}} \sim \frac{V-v}{2q\epsilon}\, e^{v/\epsilon}. \tag{3.6}$$

Proof. The condition (3.5) combined with Proposition 3.2 leads to the equation

$$\frac{\varphi_0\varphi_1 T}{\varphi_0+\varphi_1} + \left(\frac{\varphi_0-\varphi_1}{\varphi_0+\varphi_1}\right)^2 \tanh\Big((\varphi_0+\varphi_1)T/4\Big) = 1.$$

The aim is to solve it and let ϵ tend to 0. The left member in the previous equation is an increasing function of T. We introduce the change of variable $U^\epsilon = (\varphi_0+\varphi_1)T/4$. We first prove that U^ϵ increases as ϵ decreases. U^ϵ satisfies $K(U^\epsilon,\epsilon) = 1$ with

$$K(U^\epsilon,\epsilon) = \frac{4\varphi_0\varphi_1 U^\epsilon}{(\varphi_0+\varphi_1)^2} + \left(\frac{\varphi_0-\varphi_1}{\varphi_0+\varphi_1}\right)^2 \tanh(U^\epsilon).$$

Both functions $\epsilon \mapsto K(\cdot,\epsilon)$ and $x \mapsto K(x,\cdot)$ decrease for ϵ small and x large enough. It follows that U^ϵ increases when ϵ decreases and tends to 0. Let us assume $U^\epsilon \to U_0 < \infty$ in the limit $\epsilon \to 0$. Then $K(U^\epsilon, \epsilon) \to \tanh(U_0)$ which contradicts the identity $K(U^\epsilon, \epsilon) = 1$. We deduce that $U^\epsilon \to \infty$ when $\epsilon \to 0$. Now let us set $t = e^{-V/\epsilon}$ and $\beta = v/V < 1$. With these new parameters K can be written like

$$1 = \tilde{K}(U,t) = \frac{4pqt^{1+\beta}U^\epsilon}{(pt+qt^\beta)^2} + \left(\frac{pt - qt^\beta}{pt+qt^\beta}\right)^2 \tanh(U^\epsilon).$$

Let us define $\tanh(U^\epsilon) =: 1 - W$ then $U^\epsilon = \frac{1}{2}\log\left(\frac{2-W}{W}\right)$, we obtain that W tends to 0 when $t \to 0$ and the previous equation becomes:

$$1 = \hat{K}(W,t) = \frac{4pqt^{1+\beta}}{(pt+qt^\beta)^2}\log\left(\frac{2-W}{W}\right) + \left(\frac{pt-qt^\beta}{pt+qt^\beta}\right)^2 (1-W).$$

Thus, when $t \to 0$, we have

$$\begin{aligned}
\hat{K}(W,t) - 1 &= \frac{-2pqt^{1+\beta}}{(pt+qt^\beta)^2}(\log W + o(\log W)) \\
&\quad + (1 - \frac{4p}{q}t^{1-\beta} + o\left(t^{1-\beta}\right))(1-W) - 1 \\
&= \frac{-2pqt^{1+\beta}}{(pt+qt^\beta)^2}\log W - W + o(t^{1-\beta}\log W) = 0.
\end{aligned}$$

If $W = r_0 t^\alpha \log(t) R(t)$ with $\alpha = 1-\beta$ and $r_0 = -\frac{2p\alpha}{q} = -\frac{2p}{q}(1-\beta)$, we obtain the limit $R(t) \to 1$ when $t \to 0$ and therefore

$$U^\epsilon \sim -\frac{1}{2}\log\left(\frac{2p}{q}(1-\beta)t^{1-\beta}(-\log t)\right) \sim -\frac{1-\beta}{2}\log t \sim \frac{(1-\beta)V}{2\epsilon} = \frac{V-v}{2\epsilon}.$$

We recall $U^\epsilon = (\varphi_0 + \varphi_1)T/4$ which leads to the result set. □

In [10] and [18], several quality measures have been proposed to point out the optimal tuning of Y: the spectral power amplification (SPA), the SPA to noise intensity ratio (SPN), the energy (En), the energy to noise intensity ratio (ENR), the *out-of-phase* measure which describes the time spent in the most attractive state, the entropy or relative entropy. In his PhD report, I. Pavlyukevich computes for each measure the optimal relation between ϵ and T^ϵ_{mes}, the length of the period, in the small ϵ limit, we adopt a similar procedure in Proposition 3.3.

3.2 Infinitesimal generator with constant trace

Let us finally present a second example of periodic forcing in the stochastic resonance framework. This model was introduced by Eckmann and Thomas [7]. The aim in this paragraph is to find the optimal tuning between the noise intensity in the system and the period length in order to reach an average number of transitions during one period close to 1. This approach is different from the study presented in [7].

The model consists in a continuous-time Markov chain with periodic forcing: the transition rates are given by

$$\varphi_{1,2}(t) = \epsilon(a + \cos\omega t) \quad \text{and} \quad \varphi_{2,1}(t) = \epsilon(a - \cos\omega t), \quad a > 1. \tag{3.7}$$

The period satisfies $T = (2\pi)/\omega$. In this particular case, the trace of the infinitesimal generator, defined by (0.1), is a constant function. It is then quite simple to compute explicitely the *periodic stationary probability measure* and the mean number of transition.

Lemma 3.4. *The periodic stationary probability measure of the periodic forced Markov chain is given by*

$$\mu_1(t) = \frac{1}{2} - \frac{\epsilon}{4a^2\epsilon^2 + \omega^2}(2a\epsilon \cos \omega t + \omega \sin \omega t). \tag{3.8}$$

Proof. Using Proposition 1.3, we obtain

$$\mu_1(t) = \mu_1(0)e^{-2\epsilon at} + \int_0^t (\epsilon a - \epsilon \cos \omega s)e^{-2\epsilon a(t-s)} ds.$$

Hence

$$\mu_1(t) = \mu_1(0)e^{-2\epsilon at} + \frac{1 - e^{-2\epsilon at}}{2} + \frac{2\epsilon^2 ae^{-2\epsilon at} - 2\epsilon^2 a \cos \omega t - \epsilon\omega \sin \omega t}{4\epsilon^2 a^2 + \omega^2}.$$

Setting $\mu_1(T) = \mu_1(0)$, we obtain $\mu_1(0) = \dfrac{1}{2} - \dfrac{2a\epsilon^2}{4\epsilon^2 a^2 + \omega^2}$ and consequently the announced statement. $\square$

An application of Corollary 2.3 permits to describe the large time asymptotics for the first moments of the transitions from state s_1 to state s_2. It suffices to compute explicitly $\int_0^T \varphi_{1,2}(t)\mu_1(t)dt$. The result is described in the following statement while the proof is left to the reader.

Proposition 3.5. *The the mean number of transition pro period is equal to*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}_\mu[\mathcal{N}_{nT}] = \frac{\epsilon a T}{2} - \frac{\varepsilon^3 a T}{4\epsilon^2 a^2 + \omega^2}, \tag{3.9}$$

and μ *given by* (3.8).

Let us now discuss the suitable choice of the period such that $\mathbb{E}_\mu[\mathcal{N}_T] = 1$. We then need to solve

$$\pi\epsilon a(4\epsilon^2 a^2 + \omega^2) - 2\pi\epsilon^3 a = \omega(4\epsilon^2 a^2 + \omega^2). \tag{3.10}$$

It is obvious that ω is of the order ϵ, we set $\omega = \mu\epsilon$ and look for the best choice of the parameter μ. Considering (3.10), the optimal value μ is in fact a real root of the following polynomial function

$$P(\mu) := \mu^3 - \pi a\mu^2 + 4a^2\mu + 2\pi a(1 - 2a^2)$$

It is straightforward to prove that this polynomial function has a single positive root since it is increasing and verifies $P(0) < 0$. Using the Cardan formula, we can obtain an explicit expression of μ_{optimal} which depends of course on the coefficient a, this dependence is asymptotically linear as a becomes large.

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