From persistent random walks to the telegraph noise

Samuel Herrmann and Pierre Vallois
Institut de Mathématiques Elie Cartan - UMR 7502
Nancy-Université, CNRS, INRIA
B.P. 239, 54506 Vandoeuvre-lès-Nancy Cedex, France
{herrmann,vallois}@iecn.u-nancy.fr

September 29, 2008

Abstract

We study a family of memory-based persistent random walks and we prove weak convergences after space-time rescaling. The limit processes are not only Brownian motions with drift. We have obtained a continuous but non-Markov process (Z_t) which can be easely expressed in terms of a counting process (N_t). In a particular case the counting process is a Poisson process, and (Z_t) permits to represent the solution of the telegraph equation. We study in detail the Markov process $((Z_t, N_t);\ t \geq 0)$.

1 The setting of persistent random walks.

1) The simplest way to present and define a persistent random walk with value in $\mathbb{Z}$ is to introduce the process of its increments $(Y_t,\ t \in \mathbb{N})$. In the classical symmetric random walk case, this process is just a sequence of independent random variables satisfying $\mathbb{P}(Y_t = 1) = \mathbb{P}(Y_t = -1) = \frac{1}{2}$ for any $t \geq 0$. Here we shall introduce some short range memory in these increments in order to create the persistence phenomenon. Namely (Y_t) is a $\{-1, 1\}$-valued Markov chain: the law of Y_{t+1} given $\mathcal{F}_t = \sigma(Y_0, Y_1, \dots, Y_t)$ depends only on the value of Y_t. This dependence is represented by the transition probability $\pi(x, y) = \mathbb{P}(Y_{t+1} = y | Y_t = x)$ with $(x, y) \in \{-1, 1\}^2$:

$$\pi = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \qquad 0 < \alpha < 1, \quad 0 < \beta < 1.$$

The persistent random walk is the corresponding process of partial sums:

$$X_t = \sum_{i=0}^{t} Y_i \quad \text{with} \quad X_0 = Y_0 = 1 \text{ or } -1. \tag{1.1}$$

Let us discuss two particular cases:

- If $\alpha + \beta = 1$, then increments are independent and therefore the short range memory disappears. $(X_t,\ t \in \mathbb{N})$ is a classical Bernoulli random walk.
- The symmetric case $\alpha = \beta$ was historically suggested by Fürth [7] and precisely defined by Taylor [14]. Goldstein [8] developed the calculation of the random walk law and clarified the link between this process and the so-called telegraph equation. Some nice presentation of these results can be found in Weiss' book [17] and [18]. This particular short memory process is often called either *persistent or correlated random walk* or *Kac walks* (see, for instance, [5]). An interesting presentation of different limiting distributions for this correlated random walk has been given by Renshaw and Henderson [11].

2) Recently, Vallois and Tapiero [15] studied the influence of the persistence phenomenon on the first and second moments of a counting process whose increments takes their values in

$\{0,1\}$ instead of $\{-1,1\}$. They obtained some nearly linear behaviour for the expectation. Using the transformation $y \to 2y-1$, it is easy to deduce that, in our setting, we have:

$$\mathbb{E}_{-1}[X_t] := \mathbb{E}[X_t | X_0 = Y_0 = -1] = \frac{\alpha-\beta}{1-\rho}(t+1) - \frac{2\alpha}{(1-\rho)^2}(1-\rho^{t+1}). \tag{1.2}$$

$$\mathbb{E}_{+1}[X_t] := \mathbb{E}[X_t | X_0 = Y_0 = +1] = \frac{\alpha-\beta}{1-\rho}(t+1) - \frac{2\beta}{(1-\rho)^2}(1-\rho^{t+1}). \tag{1.3}$$

An application to insurance has been given in [16].
It is actually possible to determine the moment generating function (see Proposition 6.4 in Section 6).

$$\Phi(\lambda, t) = \mathbb{E}[\lambda^{X_t}], \quad (\lambda \in \mathbb{R}_+^*).$$

However it seems difficult to invert this transformation; i.e. to give the law of X_t.
3) This leads us to investigate limit distributions. It is well-known that the correctly normalized symmetric random walk converges towards the Brownian motion. Let us define the time and space normalizations. Let α_0 and β_0 denote two real numbers satisfying:

$$0 \le \alpha_0 \le 1, \quad 0 \le \beta_0 \le 1. \tag{1.4}$$

Let Δ_x be a positive small parameter so that:

$$0 \le \alpha_0 + c_0\Delta_x \le 1, \quad 0 \le \beta_0 + c_1\Delta_x \le 1, \tag{1.5}$$

where c_0 and c_1 belong to $\mathbb{R}$ (see in subsection 6.2 the allowed range of parameters).
Let $(Y_t,\, t \in \mathbb{N})$ be a Markov chain whose transition probabilities are given by the matrix:

$$\pi^{\Delta} = \begin{pmatrix} 1-\alpha_0-c_0\Delta_x & \alpha_0+c_0\Delta_x \\ \beta_0+c_1\Delta_x & 1-\beta_0-c_1\Delta_x \end{pmatrix}. \tag{1.6}$$

Let $(X_t,\ t \in \mathbb{N})$ be the random walk associated with (Y_t) (cf. (1.1)). Define the normalized random walk $(Z_s^{\Delta},\, s \in \Delta_t\,\mathbb{N})$ by the relation:

$$Z_s^{\Delta} = \Delta_x X_{s/\Delta_t}, \quad (\Delta_t > 0,\ \Delta_x > 0). \tag{1.7}$$

Set $(\tilde{Z}_s^{\Delta},\, s \ge 0)$ the continuous time process obtained by linear interpolation of (Z_s^{Δ}).
We introduce two essential parameters:

$$\rho_0 = 1 - \alpha_0 - \beta_0 \quad \text{(the asymmetry coefficient)}, \tag{1.8}$$

$$\eta_0 = \beta_0 - \alpha_0. \tag{1.9}$$

In this paper, we will aim at showing the existence of a normalization (i.e. to express Δ_t in terms of Δ_x) which depends on α_0, β_0, so that $(\tilde{Z}_s^{\Delta})$ converges in distribution, as $\Delta_x \to 0$. Our main results and the organization of the paper will be given in Section 2.

2 The main results

2.1 Case : $\rho_0 = 1$

Obviously $\rho_0 = 1$ implies that $\alpha_0 = \beta_0 = 0$, and the transition probabilities matrix is given as

$$\pi^{\Delta} = \begin{pmatrix} 1-c_0\Delta_x & c_0\Delta_x \\ c_1\Delta_x & 1-c_1\Delta_x \end{pmatrix} \quad (c_0, c_1 > 0).$$

In order to describe the limiting process, we introduce a sequence of independent identically exponentially distributed random variables $(e_n, n \ge 1)$ with $\mathbb{E}[e_n] = 1$. We construct the following counting process:

$$N_t^{c_0,c_1} = \sum_{k \ge 1} 1_{\{\lambda_1 e_1 + \lambda_2 e_2 + \ldots + \lambda_k e_k \le t\}}, \tag{2.1}$$

where

$$\lambda_k = \begin{cases} 1/c_0 & \text{if } k \text{ is odd} \\ 1/c_1 & \text{otherwise.} \end{cases} \tag{2.2}$$

Finally we define

$$Z_t^{c_0,c_1} = \int_0^t (-1)^{N_u^{c_0,c_1}} du. \tag{2.3}$$

For simplicity of notations, in the symmetric case (i.e. $c_0 = c_1$), $N_t^{c_0}$ (resp. $Z_t^{c_0}$) will stand for $N_t^{c_0,c_0}$ (resp. $Z_t^{c_0,c_0}$). The process $(Z_t^{c_0})$ has been introduced by Stroock (in [13] p. 37). It is possible to show that if we rescale $(Z_t^{c_0})$, this process converges in distribution to the standard Brownian motion. This property has been widely generalized. For instance Bardina and Jolis [1] have given weak approximation of the Brownian sheet from a Poisson process in the plane.

Theorem 2.1. *Let $\Delta_x = \Delta_t$ and $Y_0 = X_0 = -1$. Then the interpolated persistent random walk $(\tilde{Z}_s^{\Delta},\ s \geq 0)$ converges in distribution, as $\Delta_x \to 0$, to the process $(-Z_s^{c_0,c_1},\ s \geq 0)$.*
In particular if $c_0 = c_1$, then $(N_u^{c_0})$ is the Poisson process with parameter c_0.
If $Y_0 = X_0 = 1$ then the interpolated persistent random walk $(\tilde{Z}_s^{\Delta},\ s \geq 0)$ converges in distribution, as $\Delta_x \to 0$, to the process $(Z_s^{c_1,c_0},\ s \geq 0)$.

Proof. See Section 4. □

Next, in Section 3, we investigate the process $(Z_t^{c_0,c_1}, N_t^{c_0,c_1};\ t \geq 0)$. In particular we prove that it is Markov, we determine its semigroup and the law of $(Z_t^{c_0,c_1}, N_t^{c_0,c_1})$, t being fixed. This permits to prove, when $c_0 = c_1$, the well-known relation (cf. [18], [5], [8], [9]) between the solutions of the wave equation and the telegraph equation. For this reason the process $(Z_t^{c_0,c_1})$ will be called the integrated telegraph noise (ITN for short).
We emphasize that our approach based on stochastic processes gives a better understanding of analytical properties.
We will give in Section 5 below two extensions of Theorem 2.1 to the cases where (Y_t) is
1) a Markov chain which takes its values in $\{y_1, \ldots, y_k\}$,
2) a Markov chain with order 2 and valued in $\{-1, 1\}$.

2.2 Case : $\rho_0 \neq 1$

In this case, the limit process is Markov. We shall prove two kind of convergence results. The first one corresponds to the law of large numbers and the second one looks like functional central limit theorem.
Recall that $(\tilde{Z}_t^{\Delta}, t \geq 0)$ is the linear interpolation of (Z_t^{Δ}) and ρ_0 (resp. η_0) has been defined by (1.8) (resp. (1.9)).

Theorem 2.2. *1) Suppose that $r\Delta_t = \Delta_x$ with $r > 0$. Then $\tilde{Z}_t^{\Delta}$ converges to the deterministic limit $-\frac{rt\eta_0}{1-\rho_0}$ when $\Delta_x \to 0$.*
2) Suppose that $r\Delta_t = \Delta_x^2$ with $r > 0$, then the process $(\xi_t^{\Delta},\ t \geq 0)$ defined by

$$\xi_t^{\Delta} = \tilde{Z}_t^{\Delta} + \frac{t\sqrt{r}\eta_0}{(1-\rho_0)\sqrt{\Delta_t}}$$

converges in distribution to the process $(\xi_t^0,\ t \geq 0)$, as $\Delta_x \to 0$, where

$$\xi_t^0 = 2r\Big(\frac{-\overline{\tau}}{1-\rho_0} + \frac{\eta_0\tau}{(1-\rho_0)^2}\Big)t + \sqrt{\frac{r(1+\rho_0)}{1-\rho_0}\Big(1 - \frac{\eta_0^2}{(1-\rho_0)^2}\Big)}W_t, \tag{2.4}$$

$(W_t,\ t \geq 0)$ is a one-dimensional Brownian motion, $\tau = (c_0 + c_1)/2$ and $\overline{\tau} = (c_1 - c_0)/2$.

Proof. See Section 6. □

Gruber and Schweizer have proved in [10] a weak convergence result for a large class of generalized correlated random walks. However these results and ours can be only compared in the case $\alpha_0 = \beta_0$.

Note that

$$1 - \frac{\eta_0^2}{(1-\rho_0)^2} = 0 \Longleftrightarrow \alpha_0 = 0 \quad \text{or} \quad \beta_0 = 0.$$

Suppose for instance that $\alpha_0 = 0$. Then $\beta_0, c_0 > 0$ and

$$\xi_t^\Delta = \tilde{Z}_t^\Delta + \frac{t\sqrt{r}}{\sqrt{\Delta_t}} \quad \text{and} \quad \xi_t^0 = \frac{2rc_0}{\beta_0}\,t.$$

Obviously, the diffusion coefficient of (ξ_t^0) can also cancel when $\rho_0 = -1$.
Since $\rho_0 = -1 \Longleftrightarrow \alpha_0 = \beta_0 = 1$, then $c_0, c_1 < 0$ and

$$\xi_t^\Delta = \tilde{Z}_t^\Delta \quad \text{and } \xi_t^0 = -r\overline{\tau}t.$$

This shows that, in the symmetric case (i.e. $c_0 = c_1$), we have $\xi_t^0 = 0$. This means that the normalization is not the right one since the limit is null. Changing the rescaling we can obtain a non-trivial limit.

Proposition 2.3. *Suppose $\alpha_0 = \beta_0 = 1$, $c_0 = c_1 < 0$ and $r\Delta_t = \Delta_x^3$ with $r > 0$.
The interpolated persistent walk $(\tilde{Z}_t^\Delta,\ t \geq 0)$ converges in law, as $\Delta_x \to 0$, to $(\sqrt{-rc_0}W_t,\ t \geq 0)$ where (W_t) is a standard Brownian motion.*

Proof. See subsection 6.3 □

2.3 Organization of the paper

The third section presents few properties of the process $(Z_t^{c_0,c_1},\ t \geq 0)$ which has been defined by (2.3). Theorem 2.1 will be proven in Section 4. Section 5 will be devoted to two extensions of Theorem 2.1. In subsection 6.1 we determine the generating function of X_t (recall that X_t has been defined by (1.1)). This is the main tool which permits to prove Theorem 2.2 and Proposition 2.3 (see subsections 6.2 and 6.3).

3 Properties of the integrated telegraph noise

The aim of this section is to study the two dimensional process $(Z_t^{c_0,c_1},\ N_t^{c_0,c_1};\ t \geq 0)$ introduced in (2.2) and (2.3). In the particular symmetric case $c_0 = c_1$, the study is simpler since the process $(N_t^{c_0},\ t \geq 0)$ is a Poisson process with rate c_0 ($\mathbb{E}(N_t^{c_0}) = c_0 t$) and $N_0^{c_0} = 0$. However we shall study the general case.

First, we determine in Proposition 3.1 the conditional density of $Z_t^{c_0,c_1}$ given $N_t^{c_0,c_1} = n$. As a by product we obtain the distribution of $Z_t^{c_0,c_1}$ (see Proposition 3.3). Second, we prove in Proposition 3.5 that $(Z_t^{c_0,c_1}, N_t^{c_0,c_1},\ t \geq 0)$ is Markov and we determine its semi-group. We conclude this section by showing that the solution of the telegraph equation can be expressed in terms of the associated wave equation and $(Z_t^{c_0,c_0})_{t\geq 0}$. For this reason, $(Z_t^{c_0,c_1})_{t\geq 0}$ will be called the integrated telegraph noise (ITN for short). Recall that:

$$\tau = \frac{c_0 + c_1}{2}, \qquad \overline{\tau} = \frac{c_1 - c_0}{2}. \tag{3.1}$$

Proposition 3.1. *1) $\mathbb{P}(N_t^{c_0,c_1} = 0) = e^{-tc_0}$ and given $N_t^{c_0,c_1} = 0$, we have $Z_t^{c_0,c_1} = t$.
2) The counting process takes even values with probability:*

$$\mathbb{P}(N_t^{c_0,c_1} = 2k) = \frac{(c_0c_1)^k \alpha_k(t)}{2^{2k}k!(k-1)!}\, e^{-\tau t} \quad \textit{with } \alpha_k(t) = \int_{-t}^{t} (t-z)^{k-1}(t+z)^k e^{\overline{\tau}z} dz, \tag{3.2}$$

and the conditional distribution of $Z_t^{c_0,c_1}$ is given by

$$\mathbb{P}(Z_t^{c_0,c_1} \in dz | N_t^{c_0,c_1} = 2k) = \frac{1}{\alpha_k(t)}(t-z)^{k-1}(t+z)^k e^{\overline{\tau}z} 1_{[-t,t]}(z) \quad (k \geq 1). \tag{3.3}$$

3) The counting process takes odd values with probability:

$$\mathbb{P}(N_t^{c_0,c_1} = 2k+1) = \frac{c_0^{k+1} c_1^k \tilde{\alpha}_k(t)}{2^{2k+1}(k!)^2}\, e^{-\tau t} \quad \textit{with } \tilde{\alpha}_k(t) = \int_{-t}^{t} (t-z)^k (t+z)^k e^{\overline{\tau}z} dz, \tag{3.4}$$

and the conditional distribution of $Z_t^{c_0,c_1}$ *is given by*

$$\mathbb{P}(Z_t^{c_0,c_1} \in dz | N_t^{c_0,c_1} = 2k+1) = \frac{1}{\tilde{\alpha}_k(t)}(t-z)^k(t+z)^k e^{\overline{\tau} z} 1_{[-t,t]}(z) \quad (k \geq 0). \tag{3.5}$$

Corollary 3.2. *In the particular symmetric case* $c_0 = c_1$*, the conditional density function of* $Z_t^{c_0}$ *given* $N_t^{c_0} = n$ *is the centered beta density, i.e.*

$$\text{for } n = 2k,\ k \in \mathbb{N}^* : \quad f_n(t,z) = \chi_{2k} \frac{(t+z)^k (t-z)^{k-1}}{t^{2k}} 1_{[-t,t]}(z), \tag{3.6}$$

$$\text{for } n = 2k+1,\ k \in \mathbb{N} : \quad f_n(t,z) = \chi_{2k+1} \frac{(t+z)^k (t-z)^k}{t^{2k+1}} 1_{[-t,t]}(z), \tag{3.7}$$

with

$$\chi_{2k+1} = \chi_{2k+2} = \frac{1}{2^{2k+1} B(k+1,k+1)} = \frac{(2k+1)!}{2^{2k+1}(k!)^2} \quad (k \geq 0),$$

(B is the beta function (first Euler function): $B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$*).*

Proof of Proposition 3.1. Associated with $n \geq 0$ and a bounded continuous function f, we define

$$\Delta_n(f) = \mathbb{E}\left[f(Z_t^{c_0,c_1}) 1_{\{N_t^{c_0,c_1}=n\}}\right].$$

a) When $n = 0$, we obtain

$$\Delta_0(f) = \mathbb{E}\left[f(Z_t^{c_0,c_1}) 1_{\{t < \lambda_1 e_1\}}\right].$$

If $t < \lambda_1 e_1$, then $Z_t^{c_0,c_1} = t$ and

$$\Delta_0(f) = f(t)\, \mathbb{P}(t < \lambda_1 e_1) = f(t) e^{-tc_0}.$$

b) When $n \geq 1$, using (2.1) we obtain

$$\Delta_n(f) = \mathbb{E}\left[f(Z_t^{c_0,c_1}) 1_{\{\lambda_1 e_1 + \ldots + \lambda_n e_n \leq t < \lambda_1 e_1 + \ldots + \lambda_{n+1} e_{n+1}\}}\right].$$

If $\lambda_1 e_1 + \ldots + \lambda_n e_n \leq t < \lambda_1 e_1 + \ldots + \lambda_{n+1} e_{n+1}$ then

$$\begin{aligned} Z_t^{c_0,c_1} &= \int_0^{\lambda_1 e_1} (-1)^0 du + \int_{\lambda_1 e_1}^{\lambda_1 e_1 + \lambda_2 e_2} (-1) du + \ldots + \int_{\lambda_1 e_1 + \ldots + \lambda_{n-1} e_{n-1}}^{\lambda_1 e_1 + \ldots + \lambda_n e_n} (-1)^{n-1} du \\ &+ \int_{\lambda_1 e_1 + \ldots + \lambda_n e_n}^{t} (-1)^n du. \end{aligned}$$

Hence

$$Z_t^{c_0,c_1} = \lambda_1 e_1 - \lambda_2 e_2 + \lambda_3 e_3 + \ldots + (-1)^{n-1} \lambda_n e_n + (-1)^n (t - \lambda_1 e_1 - \ldots - \lambda_n e_n). \tag{3.8}$$

c) Evaluation of $\Delta_{2k}(f)$, $k \geq 1$.
We introduce two sequences of random variables associated with (e_n):

$$\xi_k^e = e_2 + \ldots + e_{2k}, \quad \xi_k^o = e_1 + \ldots + e_{2k-1}, \quad (k \geq 1). \tag{3.9}$$

By (3.8), (2.2) and (3.9) we obtain the simpler expression

$$\Delta_{2k}(f) = \mathbb{E}\left[f(t - 2\xi_k^e / c_1) 1_{\{\xi_k^o / c_0 + \xi_k^e / c_1 \leq t < \xi_k^o / c_0 + \xi_k^e / c_1 + e_{2k+1} / c_0\}}\right].$$

Note that from our assumptions, ξ_k^e, ξ_k^o and e_{2k+1} are independent r.v.'s, ξ_k^o and ξ_k^e are both gamma distributed with parameter k. Consequently:

$$\begin{aligned} \Delta_{2k}(f) &= \frac{1}{((k-1)!)^2} \int_{D_t} \exp\{-c_0(t - y/c_0 - x/c_1)\} f(t - 2x/c_1) e^{-x-y} x^{k-1} y^{k-1} dx\, dy \\ &= \frac{c_0^k e^{-tc_0}}{k!(k-1)!} \int_0^{tc_1} f(t - 2x/c_1) x^{k-1} (t - x/c_1)^k \exp\left\{\left(\frac{c_0}{c_1} - 1\right) x\right\} dx, \end{aligned}$$

where $D_t = \mathbb{R}_+^2 \cap \{y/c_0 + x/c_1 \leq t\}$. Using the change of variable $z = t - 2x/c_1$, we obtain $x = c_1 \frac{t-z}{2}$, $t - x/c_1 = \frac{t+z}{2}$ and

$$\Delta_{2k}(f) = \frac{(c_0c_1)^k}{2} \frac{e^{-(c_0+c_1)t/2}}{k!(k-1)!} \int_{-t}^{t} f(z)\Big(\frac{t-z}{2}\Big)^{k-1}\Big(\frac{t+z}{2}\Big)^k \exp\{(c_1-c_0)z/2\}dz. \quad (3.10)$$

Finally (3.10) and (3.1) imply (3.2) and (3.3).

d) Evaluation of $\Delta_{2k+1}(f)$ for $k \geq 0$. The arguments are similar to those presented in part c). On the event $\xi_{k+1}^o/c_0 + \xi_k^e/c_1 \leq t < \xi_{k+1}^o/c_0 + \xi_k^e/c_1 + e_{2k+2}/c_1$, we have: $Z_t^{c_0,c_1} = 2\xi_{k+1}^o/c_0 - t$; this implies

$$\Delta_{2k+1}(f) = \mathbb{E}\Big[1_{\{\xi_{k+1}^o/c_0 + \xi_k^e/c_1 \leq t\}} \exp\Big(-c_1(t - \xi_{k+1}^o/c_0 - \xi_k^e/c_1)\Big) f(2\xi_{k+1}^o/c_0 - t)\Big].$$

Since ξ_{k+1}^o and ξ_k^e are independent and gamma distributed with parameter $k+1$ (resp. k), we get

$$\Delta_{2k+1}(f) = \frac{c_0^{k+1}c_1^k}{2(k!)^2} e^{-(c_0+c_1)t/2} \int_{-t}^{t} f(z)\Big(\frac{t-z}{2}\Big)^k\Big(\frac{t+z}{2}\Big)^k \exp\Big\{(c_1-c_0)z/2\Big\}dz. \quad (3.11)$$

This leads directly to (3.4) and (3.5). □

Let us recall the definition of the modified Bessel functions:

$$I_\nu(\xi) = \sum_{m \geq 0} \frac{(\xi/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}.$$

Proposition 3.3. *The distribution of $Z_t^{c_0,c_1}$ is given by*

$$\mathbb{P}(Z_t^{c_0,c_1} \in dx) = e^{-c_0t}\delta_t(dx) + e^{-\tau t} f(t,x) 1_{[-t,t]}(x), \quad (3.12)$$

where

$$f(t,x) = \frac{1}{2}\Big[\sqrt{\frac{c_0c_1(t+x)}{t-x}} I_1\Big(\sqrt{c_0c_1(t^2-x^2)}\Big) + c_0 I_0\Big(\sqrt{c_0c_1(t^2-x^2)}\Big)\Big] e^{\overline{\tau}x}. \quad (3.13)$$

Remark 3.4. *Let us focus our attention to the symmetric case $c_0 = c_1$. We can introduce some randomization of the initial condition as follows: let ϵ be a $\{-1,1\}$-valued random variable, independent from the Poisson process $N_t^{c_0}$, with $p := \mathbb{P}(\epsilon = 1) = 1 - \mathbb{P}(\epsilon = -1)$. It is easy to deduce from* (3.12) *that we have*

$$\mathbb{P}(\epsilon Z_t^{c_0}/t \in dx) = \Big(p\delta_1(dx) + (1-p)\delta_{-1}(dx) + g(t,x)dx\Big)e^{-c_0t}, \quad (3.14)$$

with

$$g(t,x) = \frac{c_0t}{2}\Big\{I_0\Big(c_0t\sqrt{1-x^2}\Big) + \frac{1+(2p-1)x}{\sqrt{1-x^2}} I_1\Big(c_0t\sqrt{1-x^2}\Big)\Big\}1_{[-1,1]}(x)$$

and $\delta_1(dx)$ (resp. $\delta_{-1}(dx)$) is the Dirac measure at 1 (resp. -1).
In the particular case $p = 1/2$, $x \to g(t,x)$ is an even function. G.H. Weiss ([18] p.393) proved (3.14) *using an analytic method based on Fourier-Laplace transform.*

Proof of Proposition 3.3. The proof is a direct consequence of the expression of Proposition 3.1. Indeed, for each bounded continuous function φ we denote

$$\Delta = \mathbb{E}[\varphi(Z_t^{c_0,c_1})] = \varphi(t)e^{-c_0t} + \sum_{k\geq 1}\Delta_{2k}(\varphi) + \sum_{k\geq 0}\Delta_{2k+1}(\varphi) = \varphi(t)e^{-c_0t} + \Delta_e + \Delta_o,$$

where $\Delta_n(\varphi) = \mathbb{E}[\varphi(Z_t^{c_0,c_1})1_{\{N_t^{c_0,c_1}=n\}}]$. Using (3.2) and (3.3) we get

$$\Delta_e = e^{-\tau t}\int_{-t}^{t} \varphi(z) S_e(z) e^{\overline{\tau}z} dz,$$

with

$$\begin{aligned} S_e(z) &= \frac{1}{2}\sum_{k\geq 1}\frac{(c_0c_1)^k}{k!(k-1)!}\Big(\frac{t-z}{2}\Big)^{k-1}\Big(\frac{t+z}{2}\Big)^k \\ &= \frac{1}{2}\sqrt{c_0c_1}\sqrt{\frac{t+z}{t-z}}\sum_{k\geq 0}\frac{1}{k!(k+1)!}\Big(\frac{\sqrt{c_0c_1(t^2-z^2)}}{2}\Big)^{2k+1} \\ &= \frac{1}{2}\sqrt{c_0c_1}\sqrt{\frac{t+z}{t-z}}I_1\Big(\sqrt{c_0c_1(t^2-z^2)}\Big). \end{aligned}$$

For the odd indexes, by (3.4) and (3.5) we get

$$\Delta_o = e^{-\tau t}\int_{-t}^{t}\varphi(z)S_o(z)e^{\overline{\tau}z}dz,$$

with

$$S_o(z) = \frac{1}{2}\sum_{k\geq 0}\frac{c_0^{k+1}c_1^k}{(k!)^2}\Big(\frac{t^2-z^2}{4}\Big)^k = \frac{c_0}{2}I_0\Big(\sqrt{c_0c_1(t^2-z^2)}\Big).$$

□

Proposition 3.5. *1)* $(Z_t^{c_0,c_1}, N_t^{c_0,c_1};\ t\geq 0)$ *is a* $\mathbb{R}\times\mathbb{N}$*-valued Markov process.*
2) Let $s\geq 0$ *and* $n\geq 0$*. Conditionally on* $Z_s^{c_0,c_1}=x$ *and* $N_s^{c_0,c_1}=n$*,* $\Big((Z_{t+s}^{c_0,c_1}, N_{t+s}^{c_0,c_1}),\ t\geq 0\Big)$ *is distributed as*

$$\begin{cases} \Big(\Big(x+\int_0^t(-1)^{N_u^{c_0,c_1}}du, n+N_t^{c_0,c_1}\Big),\ t\geq 0\Big) & \text{when } n \text{ is even,} \\ \Big(\Big(x-\int_0^t(-1)^{N_u^{c_1,c_0}}du, n+N_t^{c_1,c_0}\Big),\ t\geq 0\Big) & \text{otherwise.} \end{cases}$$

Remark 3.6. *Note that Propositions 3.5 and 3.1 permit to determine the semigroup of* $\Big((Z_t^{c_0,c_1}, N_t^{c_0,c_1}),\ t\geq 0\Big)$ *i.e.* $\mathbb{P}(Z_t^{c_0,c_1}\in dx,\ N_t^{c_0,c_1}=n|Z_s^{c_0,c_1}=y,\ N_s^{c_0,c_1}=m)$ *where* $t>s$*,* $n\geq m$ *and* $y\in[-s,s]$*.*

Proof of Proposition 3.5. Let $t>s\geq 0$. Using (2.3) we get

$$Z_t^{c_0,c_1} = Z_s^{c_0,c_1} + (-1)^{N_s^{c_0,c_1}}\int_0^{t-s}(-1)^{\tilde N_u^s}du,$$

where $\tilde N_u^s = N_{s+u}^{c_0,c_1} - N_s^{c_0,c_1}$, $u\geq 0$.
Note that $(\tilde N_u^s;\ u\geq 0)\overset{(d)}{=}(N_u^{c_0,c_1};\ u\geq 0)$ if $N_s^{c_0,c_1}\in 2\,\mathbb{N}$ and $(\tilde N_u^s;\ u\geq 0)\overset{(d)}{=}(N_u^{c_1,c_0};\ u\geq 0)$ if $N_s^{c_0,c_1}\in 2\,\mathbb{N}+1$. This shows Proposition 3.5. □

Next, we determine (in Proposition 3.8 below) the Laplace transform of the r.v. $Z_t^{c_0,c_1}$. It is possible to use the distribution of $Z_t^{c_0,c_1}$ (cf Proposition 3.3), but this method has the disadvantage of leading to heavy calculations. We develop here a method which uses the fact that $(Z_s^{c_0,c_1};\ s\geq 0)$ is a stochastic process given by (2.3). The key tool is Lemma 3.7 below. Roughly speaking Lemma 3.7 gives the generator of the Markov process $(Z_t^{c_0,c_1}, N_t^{c_0,c_1})$. Lemma 3.7 is an important ingredient in the proof of Proposition 3.11 besides.

Lemma 3.7. *Let* $F:\mathbb{R}\times\mathbb{N}\to\mathbb{R}$ *denote a bounded and continuous function such that* $z\to F(z,n)$ *is of class* $\mathcal{C}^1$ *for all* n*. Then*

$$\begin{aligned} \frac{d}{dt}\,\mathbb{E}[F(Z_t^{c_0,c_1}, N_t^{c_0,c_1})] &= \mathbb{E}\Big[\ \frac{\partial F}{\partial z}(Z_t^{c_0,c_1}, N_t^{c_0,c_1})(-1)^{N_t^{c_0,c_1}}\Big] \\ &+\mathbb{E}\Big[\ \Big(F(Z_t^{c_0,c_1}, N_t^{c_0,c_1}+1) - F(Z_t^{c_0,c_1}, N_t^{c_0,c_1})\Big) \\ &\quad\times\Big(c_1 1_{\{N_t^{c_0,c_1}\in 2\,\mathbb{N}+1\}} + c_0 1_{\{N_t^{c_0,c_1}\in 2\,\mathbb{N}\}}\Big)\Big]. \end{aligned} \tag{3.15}$$

Proof. Let us denote by $\Delta(t)=\mathbb{E}[F(Z_t^{c_0,c_1}, N_t^{c_0,c_1})]$. In order to compute the t-derivative we shall decompose the increment of $t\to\Delta(t)$ in a sum of two terms:

$$\frac{\Delta(t+h)-\Delta(t)}{h} = B_h + C_h,$$

with

$$B_h = \frac{1}{h}\Big\{ \mathbb{E}[F(Z^{c_0,c_1}_{t+h}, N^{c_0,c_1}_{t+h})] - \mathbb{E}[F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t+h})]\Big\},$$

$$C_h = \frac{1}{h}\Big\{ \mathbb{E}[F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t+h})] - \mathbb{E}[F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t})]\Big\}.$$

Since $F(\cdot, n)$ is continuously differentiable with respect to the variable z and $t \to Z^{c_0,c_1}_t$ is differentiable (cf (2.3)), using the change of variable formula we obtain

$$\frac{1}{h}\Big\{F(Z^{c_0,c_1}_{t+h}, N^{c_0,c_1}_{t+h}) - F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t+h})\Big\} = \frac{1}{h}\int_t^{t+h} \frac{\partial F}{\partial z}(Z^{c_0,c_1}_{u}, N^{c_0,c_1}_{t+h})(-1)^{N^{c_0,c_1}_u}\,du.$$

Therefore

$$\lim_{h\to 0} B_h = \mathbb{E}\Big[\frac{\partial F}{\partial z}(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t})(-1)^{N^{c_0,c_1}_t}\Big]. \tag{3.16}$$

In order to study the limit of C_h, we consider two cases: $N^{c_0,c_1}_t \in 2\,\mathbb{N}$ and $N^{c_0,c_1}_t \in 2\,\mathbb{N}+1$:

$$\begin{aligned} C_h &= \frac{1}{h}\,\mathbb{E}\Big[\Big(F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t} + \tilde{N}^{c_1,c_0}_h) - F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t})\Big)1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}+1\}}\Big] \\ &+ \frac{1}{h}\,\mathbb{E}\Big[\Big(F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t} + \tilde{N}^{c_0,c_1}_h) - F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t})\Big)1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}\}}\Big], \end{aligned}$$

where $\tilde{N}_h = N^{c_0,c_1}_{t+h} - N^{c_0,c_1}_t$.

According to Proposition 3.5, conditionally on Z^{c_0,c_1}_t and $N^{c_0,c_1}_t \in 2\,\mathbb{N}$ (resp. $N^{c_0,c_1}_t \in 2\,\mathbb{N}+1$), $\tilde{N}_h$ is distributed as N^{c_0,c_1}_h (resp. N^{c_1,c_0}_h). Note that Proposition 3.1 implies that $\mathbb{P}(N^{c_0,c_1}_h \geq 2) = o(h)$ and

$$\mathbb{P}(N^{c_0,c_1}_h = 1) = \frac{c_0}{2}\Big(\frac{e^{\overline{\tau}h} - e^{-\overline{\tau}h}}{\overline{\tau}}\Big)e^{-\tau h} = c_0 h + o(h).$$

Consequently

$$\begin{aligned} \lim_{h\to 0} C_h &= c_1\,\mathbb{E}\Big[\Big(F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t} + 1) - F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t})\Big)1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}+1\}}\Big] \\ &+ c_0\,\mathbb{E}\Big[\Big(F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t} + 1) - F(Z^{c_0,c_1}_{t}, N^{c_0,c_1}_{t})\Big)1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}\}}\Big]. \end{aligned} \tag{3.17}$$

Then, (3.16) and (3.17) clearly imply Lemma 3.7. □

Let us introduce the two quantities:

$$L_e(t) = \mathbb{E}\Big[e^{-\mu Z^{c_0,c_1}_t} 1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}\}}\Big] \text{ and } L_o(t) = \mathbb{E}\Big[e^{-\mu Z^{c_0,c_1}_t} 1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}+1\}}\Big],\ (t \geq 0, \mu \in \mathbb{R}). \tag{3.18}$$

Since $|Z^{c_0,c_1}_t| \leq t$, then $L_e(t)$ and $L_o(t)$ are well defined for any $\mu \in \mathbb{R}$. Note that $\mu \to L_e(t)$ (resp. $\mu \to L_o(t)$) is a Laplace transform. We have mentioned the t-dependency only because it will play an important role in our proof of Proposition 3.8 below.

Proposition 3.8. *Let $L_e(t)$ and $L_o(t)$ be defined by* (3.18)*. Then*

$$L_e(t) = \frac{1}{\sqrt{\mathcal{E}}}\Big((-\mu + \overline{\tau})\sinh(t\sqrt{\mathcal{E}}) + \sqrt{\mathcal{E}}\cosh(t\sqrt{\mathcal{E}})\Big)e^{-\tau t}, \tag{3.19}$$

$$L_o(t) = \frac{c_0}{\sqrt{\mathcal{E}}}\sinh(t\sqrt{\mathcal{E}})e^{-\tau t}, \tag{3.20}$$

$$\mathbb{E}[e^{-\mu Z^{c_0,c_1}_t}] = \frac{1}{\sqrt{\mathcal{E}}}\Big[(-\mu + \tau)\sinh(t\sqrt{\mathcal{E}}) + \sqrt{\mathcal{E}}\cosh(t\sqrt{\mathcal{E}})\Big]e^{-\tau t}, \tag{3.21}$$

where $\mathcal{E} = \mu^2 - 2\overline{\tau}\mu + \tau^2$.

Proof. Applying Lemma 3.7 with the particular function $F(z, n) = e^{-\mu z}1_{\{n \in 2\,\mathbb{N}\}}$, we have:

$$\begin{aligned} \frac{d}{dt}L_e(t) &= -\mu\,\mathbb{E}\Big[e^{-\mu Z^{c_0,c_1}_t}(-1)^{N^{c_0,c_1}_t}1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}\}}\Big] \\ &+ E\Big[e^{-\mu Z^{c_0,c_1}_t}\Big(1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}+1\}} - 1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}\}}\Big) \\ &\quad \times\Big(c_1 1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}+1\}} + c_0 1_{\{N^{c_0,c_1}_t \in 2\,\mathbb{N}\}}\Big)\Big] \end{aligned}$$

We deduce

$$\frac{d}{dt}L_e(t) = -(\mu + c_0)L_e(t) + c_1 L_o(t).$$

Similarly

$$\begin{aligned}\frac{d}{dt}L_o(t) &= -\mu\,\mathbb{E}\left[e^{-\mu Z_t^{c_0,c_1}}(-1)^{N_t^{c_0,c_1}}1_{\{N_t^{c_0,c_1}\in 2\,\mathbb{N}+1\}}\right]\\ &+ E\Big[e^{-\mu Z_t^{c_0,c_1}}\Big(1_{\{N_t^{c_0,c_1}\in 2\,\mathbb{N}\}} - 1_{\{N_t^{c_0,c_1}\in 2\,\mathbb{N}+1\}}\Big)\\ &\quad\times\Big(c_1 1_{\{N_t^{c_0,c_1}\in 2\,\mathbb{N}+1\}} + c_0 1_{\{N_t^{c_0,c_1}\in 2\,\mathbb{N}\}}\Big)\Big].\end{aligned}$$

We get therefore

$$\frac{d}{dt}L_o(t) = (\mu - c_1)L_o(t) + c_0 L_e(t).$$

To sum up

$$\frac{d}{dt}\begin{pmatrix} L_e(t)\\ L_o(t)\end{pmatrix} = \begin{pmatrix} -\mu - c_0 & c_1\\ c_0 & \mu - c_1\end{pmatrix}\begin{pmatrix} L_e(t)\\ L_o(t)\end{pmatrix}.$$

We deduce the expressions of $L_e(t)$ and $L_o(t)$:

$$L_e(t) = a_+e^{\lambda_+ t} + a_-e^{\lambda_- t} \qquad L_o(t) = b_+e^{\lambda_+ t} + b_-e^{\lambda_- t}, \tag{3.22}$$

where $\lambda_\pm = -\tau \pm \sqrt{\mu^2 - 2\overline{\tau}\mu + \tau^2} = -\tau \pm \sqrt{\mathcal{E}}$.
The constants $a_\pm$ and $b_\pm$ are evaluated with the initial conditions:

$$L_e(0) = \mathbb{P}(N_0^{c_0,c_1} \in 2\,\mathbb{N}) = 1, \qquad L_o(0) = \mathbb{P}(N_0^{c_0,c_1} \in 2\,\mathbb{N}+1) = 0,$$

$$\frac{dL_e}{dt}(0) = -(\mu + c_0)L_e(0) + c_1L_o(0) = -\mu - c_0,$$

$$\frac{dL_o}{dt}(0) = (\mu - c_1)L_o(0) + c_0L_e(0) = c_0.$$

We obtain

$$a_+ = \frac{1}{2\sqrt{\mathcal{E}}}(-\mu + \overline{\tau} + \sqrt{\mathcal{E}}) \quad \text{and} \quad a_- = \frac{1}{2\sqrt{\mathcal{E}}}(\mu - \overline{\tau} + \sqrt{\mathcal{E}}), \tag{3.23}$$

$$b_+ = \frac{c_0}{2\sqrt{\mathcal{E}}} \quad \text{and} \quad b_- = -\frac{c_0}{2\sqrt{\mathcal{E}}} \tag{3.24}$$

Using (3.22), (3.23) and (3.24), Proposition 3.8 follows. □

It is easy to deduce two direct consequences of Proposition 3.8. First, taking $\mu = 0$ we obtain $\mathbb{P}(N_t^{c_0,c_1} \in 2\,\mathbb{N})$ and $\mathbb{P}(N_t^{c_0,c_1} \in 2\,\mathbb{N}+1)$. Second, taking the expectation in (2.3) we get the mean of $Z_t^{c_0,c_1}$.

Corollary 3.9. *We have:*

$$\mathbb{P}(N_t^{c_0,c_1} \in 2\,\mathbb{N}) = \frac{1}{\tau}\Big[\overline{\tau}\sinh(\tau t) + \tau\cosh(\tau t)\Big]e^{-\tau t},$$

$$\mathbb{P}(N_t^{c_0,c_1} \in 2\,\mathbb{N}+1) = \frac{c_0}{\tau}\sinh(\tau t)e^{-\tau t},$$

and

$$\mathbb{E}[Z_t^{c_0,c_1}] = \frac{\overline{\tau}}{\tau}t + \frac{c_0}{2\tau^2}(1 - e^{-2\tau t}).$$

Remark 3.10. *The Laplace transform with respect to the time variable can also be explicitly computed. We define $F(\mu,s) = \int_0^\infty e^{-st}\,\mathbb{E}[e^{-\mu Z_t^{c_0,c_1}}]dt$. Integrating (3.21) with respect to dt we get*

$$\begin{aligned}F(\mu,s) &= \frac{1}{2\sqrt{\mathcal{E}}}\Big(\sqrt{\mathcal{E}} + (-\mu+\tau)\Big)\frac{1}{s - \sqrt{\mathcal{E}} + \tau} + \frac{1}{2\sqrt{\mathcal{E}}}\Big(\sqrt{\mathcal{E}} - (-\mu+\tau)\Big)\frac{1}{s + \sqrt{\mathcal{E}} + \tau}\\ &= \frac{(\sqrt{\mathcal{E}} - \mu + \tau)(s + \sqrt{\mathcal{E}} + \tau) + (\sqrt{\mathcal{E}} + \mu - \tau)(s - \sqrt{\mathcal{E}} + \tau)}{2\sqrt{\mathcal{E}}((s+\tau)^2 - \mathcal{E})}\\ &= \frac{2s\sqrt{\mathcal{E}} + 4\tau\sqrt{\mathcal{E}} - 2\mu\sqrt{\mathcal{E}}}{2\sqrt{\mathcal{E}}((s+\tau)^2 - \mathcal{E})} = \frac{s + 2\tau - \mu}{(s+\tau)^2 - \mathcal{E}}\end{aligned}$$

In the symmetric case, $\mathcal{E}$ equals $\mu^2 + c_0^2$, then

$$F(\mu, s) = \frac{s + 2c_0 - \mu}{s^2 + 2sc_0 - \mu^2}. \tag{3.25}$$

Let (Z_t) be the symmetrization of $(Z_t^{c_0})$ which is defined by an initial randomization:

$$Z_t = \epsilon Z_t^{c_0}, \quad t \geq 0,$$

where ϵ is independent of $Z_t^{c_0}$ and $\mathbb{P}(\epsilon = \pm 1) = 1/2$.
Relation (3.25) *implies*

$$\int_0^\infty e^{-st}\, \mathbb{E}[e^{-\mu Z_t}] dt = \frac{s + 2c_0}{s^2 + 2sc_0 - \mu^2}.$$

This identity has been obtained by Weiss in [18].

Let us now present a link between the ITN process and the telegraph equation in the particular symmetric case $c_0 = c_1 = c > 0$. Recall that (N_t^c) is a Poisson process with parameter c.
Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ whose first and second derivatives are bounded. We define

$$u(x,t) = \frac{1}{2}\Big\{ f(x + at) + f(x - at) \Big\}, \quad x \in \mathbb{R},\ t \geq 0.$$

Then (cf [5]) u is the unique solution of the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = f(x), \quad \dfrac{\partial u}{\partial t}(x,0) = 0. \end{cases}$$

Proposition 3.11. *The function*

$$w(x,t) = \mathbb{E}\left[u\Big(x, \int_0^t (-1)^{N_s^c} ds\Big)\right], \quad (x \in \mathbb{R}, t \geq 0)$$

is the solution of the telegraph equation (TE)

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} + 2c \dfrac{\partial w}{\partial t} = a^2 \dfrac{\partial^2 w}{\partial x^2}, \\ w(x,0) = f(x), \quad \dfrac{\partial w}{\partial t}(x,0) = 0. \end{cases}$$

This result can be proved using asymptotic analysis applied to difference equation associated with the persistent random walk [8] or using Fourier transforms [18]. Here we shall present a new proof.
Proof of Proposition 3.11. Applying twice Lemma 3.7 to $(z,n) \to u(x,z)$ and $(z,n) \to \frac{\partial u}{\partial t}(x,z)(-1)^n$ we obtain:

$$\frac{\partial w}{\partial t}(x,t) = \mathbb{E}\left[\frac{\partial u}{\partial t}\Big(x, \int_0^t (-1)^{N_s^c} ds\Big)(-1)^{N_t^c}\right].$$

and

$$\frac{\partial^2 w}{\partial t^2}(x,t) = \mathbb{E}\left[\frac{\partial^2 u}{\partial t^2}\Big(x, \int_0^t (-1)^{N_s^c} ds\Big)\right] - 2c\, \mathbb{E}\left[\frac{\partial u}{\partial t}\Big(x, \int_0^t (-1)^{N_s^c} ds\Big)(-1)^{N_t^c}\right].$$

Since u solves the wave equation we have

$$\frac{\partial^2 w}{\partial t^2}(x,t) = a^2 \frac{\partial^2 w}{\partial x^2}(x,t) - 2c \frac{\partial w}{\partial t}(x,t).$$

The function w is actually the solution of the telegraph equation. It is easy to prove that w satisfies the boundary conditions. □

Let us note that Proposition 3.11 can be extended to the asymmetric case $c_0 \neq c_1$. In this general case the telegraph equation is replaced by a linear system of partial differential equations.

Remark 3.12. *1) In [5], [9], an extension of Proposition 3.11 has been proved. Let A be the generator of a strongly continuous group of bounded linear operators on a Banach space. If w is the unique solution of this abstract "wave equation":*

$$\frac{\partial^2 w}{\partial t^2} = A^2 w;\ w(\cdot,0) = f,\ \frac{\partial w}{\partial t}(\cdot,0) = Ag \quad (f,g \in \mathcal{D}(A))$$

then $u(x,t) = \mathbb{E}\left[w\Big(x, \int_0^t (-1)^{N_s^c} ds\Big)\right]$ *solves the abstract "telegraph equation":*

$$\frac{\partial^2 u}{\partial t^2} = A^2 u - 2c\frac{\partial u}{\partial t},\quad u(\cdot,0) = f,\quad \frac{\partial u}{\partial t}(\cdot,0) = Ag.$$

2) In the same vein as [9], Enriquez [6] has introduced processes with jumps to represent solutions of some linear differential equations and biharmonic equations in the presence of a potential term. Moreover useful references are given in [6].
3) It is easy to deduce from Lemma 3.7 that the functions

$$w_e(x,t) = \mathbb{E}\left[u\Big(x, \int_0^t (-1)^{N_s^{c_0,c_1}} ds\Big) 1_{\{N_t^{c_0,c_1} \in 2\,\mathbb{N}\}}\right], \quad (x \in \mathbb{R}, t \geq 0)$$

$$w_o(x,t) = \mathbb{E}\left[u\Big(x, \int_0^t (-1)^{N_s^{c_0,c_1}} ds\Big) 1_{\{N_t^{c_0,c_1} \in 2\,\mathbb{N}+1\}}\right], \quad (x \in \mathbb{R}, t \geq 0)$$

are solutions of the general telegraph system (TS)

$$\begin{cases} \dfrac{\partial^2 w_e}{\partial t^2} = (c_0c_1 - c_0^2)w_e + (c_0c_1 - c_1^2)w_o - 2c_0\dfrac{\partial w_e}{\partial t} + a^2\dfrac{\partial^2 w_e}{\partial x^2}, \\ \dfrac{\partial^2 w_o}{\partial t^2} = (c_0c_1 - c_0^2)w_e + (c_0c_1 - c_1^2)w_o - 2c_1\dfrac{\partial w_o}{\partial t} + a^2\dfrac{\partial^2 w_o}{\partial x^2}, \\ w_e(x,0) = f(x), \quad w_o(x,0) = 0 \quad \dfrac{\partial w_e}{\partial t}(x,0) = -c_0 f(x) \quad \dfrac{\partial w_o}{\partial t}(x,0) = c_0 f(x). \end{cases}$$

4 Convergence of the persistent walk to the ITN

Suppose $\rho_0 = 1$. The aim of this section is to prove the convergence of the interpolated persistent random walk towards the generalized integrated telegraph noise (ITN) i.e. Theorem 2.1. Let us start with preliminary results.
First, let us recall that $(X_n, n \in \mathbb{N})$ is the persistent random walk starting in 0 defined by the increments process $(Y_n,\ n \in \mathbb{N})$ (see Section 1) with transition probabilities

$$\pi^\Delta = \begin{pmatrix} 1 - c_0\Delta_t & c_0\Delta_t \\ c_1\Delta_t & 1 - c_1\Delta_t \end{pmatrix}.$$

Let $(T_k;\ k \geq 1)$ be the sign changes sequence of times :

$$\begin{cases} T_1 = \inf\{t \geq 1\ : Y_t \neq Y_0\} \\ T_{k+1} = \inf\{t > T_k\ : Y_t \neq Y_{T_k}\}; \quad k \geq 1. \end{cases} \tag{4.1}$$

We put $T_0 = 0$ and

$$A_k = T_k - T_{k-1} \quad k \geq 1 \tag{4.2}$$

Let N_t be the number of times over $[0,t]$ so that the sign of (Y_n) changes:

$$N_t = \sum_{j \geq 1} 1_{\{T_j \leq t\}} \tag{4.3}$$

The definition of N_t implies that:

$$N_t = k \Longleftrightarrow T_k \leq t < T_{k+1}$$

We suppose in this subsection that $Y_0 = -1$.
We deduce from the identities above:

$$X_t = \sum_{j=1}^{k} (-1)^j A_j + (-1)^{k+1}(t - T_k + 1) \quad \text{where } k = N_t. \tag{4.4}$$

By (4.2) we obtain:

$$T_k = A_1 + \ldots + A_k \quad k \geq 1. \tag{4.5}$$

Hence the equations (4.3), (4.4) and (4.5) permit to emphasize the bijective correspondence between $(X_n;\ n \in \mathbb{N})$ and $(A_k;\ k \in \mathbb{N})$.
We introduce the normalization of $(X_n;\ n \in \mathbb{N})$ given by (1.7) with $\Delta_x = \Delta_t$:

$$Z_s^{\Delta} = \Delta_t X_{s/\Delta_t} \quad (s/\Delta_t \in \mathbb{N}). \tag{4.6}$$

Let us define:

$$N_s^{\Delta} = \sum_{j \geq 1} 1_{\{\sum_{i=1}^{j} \Delta_t A_j \leq s\}} \quad s \geq 0. \tag{4.7}$$

Let us note that

$$N_s^{\Delta} = N_{s/\Delta_t} \quad \text{if } s/\Delta_t \in \mathbb{N}\,.$$

That permits to extend the definition of Z_s^{Δ} to any $s \geq 0$ by setting

$$\tilde{Z}_s^{\Delta} = \sum_{j=1}^{k} (-1)^j (\Delta_t A_j) + (-1)^{k+1}(s - \Delta_t T_k + \Delta_t) \quad k = N_s^{\Delta}. \tag{4.8}$$

Obviously $\tilde{Z}_s^{\Delta} = Z_s^{\Delta}$ if $s/\Delta_t \in \mathbb{N}$.
In order to study the asymptotic behaviour of $(\tilde{Z}_s^{\Delta})$ as $\Delta_t \to 0$, we shall first prove the convergence in distribution of $(\Delta_t A_j)_{j\geq 1}$ and $(N_s^{\Delta})_{s \geq 0}$.
We recall that some random variable ξ is exponentially distributed with parameter $\lambda > 0$ if its density is given by $\frac{1}{\lambda} e^{-x/\lambda} 1_{\{x \geq 0\}}$.

Lemma 4.1. *The random variables (A_k) are independent and $\Delta_t A_{2k}$ (resp. $\Delta_t A_{2k+1}$) converges in distribution, as $\Delta_t \to 0$, to the exponential law with parameter $\frac{1}{c_1}$ (resp. $\frac{1}{c_0}$).*

Proof. Since (Y_n) is a Markov chain, then the (A_k) are independent. First let us study the convergence in distribution of the sequence $\Delta_t A_{2k}$. We use the Laplace transform of $\Delta_t A_{2k}$: $\varphi(\mu) = \mathbb{E}[e^{-\mu \Delta_t A_{2k}}]$, $\mu \geq 0$. Since A_{2k} is geometrically distributed with parameter $c_1 \Delta_t$, we obtain

$$\begin{aligned} \varphi(\mu) &= \sum_{j=1}^{\infty} e^{-\mu \Delta_t j}(1 - c_1 \Delta_t)^{j-1} c_1 \Delta_t \\ &= \frac{c_1 \Delta_t}{e^{\mu \Delta_t} - (1 - c_1 \Delta_t)} = \frac{c_1 \Delta_t}{(\mu + c_1)\Delta_t + o(\Delta_t)} = \frac{c_1}{\mu + c_1} + o(\Delta_t) \end{aligned} \tag{4.9}$$

The function $\varphi(\mu)$ converges for any $\mu \geq 0$ to the Laplace transform of some exponential law with parameter c_1^{-1}. This proves the convergence in distribution of $\Delta_t A_{2k}$. Concerning A_{2k-1} the arguments are similar. □

Let us recall that the counting process $(N_t^{c_0,c_1},\ t \geq 0)$ has been defined through the sequence of jumps $(e_n; n \geq 1)$ via (2.1), and $(e_n;\ n \geq 1)$ are i.i.d. and exponentially distributed.

Lemma 4.2. *Let $s > 0$, $k \geq 1$ and $\Phi_k : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function. Then*

1) $\lim_{\Delta_t \to 0} \mathbb{P}(N_s^{\Delta} = 0) = \mathbb{P}(N_s^{c_0,c_1} = 0)$

2) $\lim_{\Delta_t \to 0} \mathbb{E}[\Phi_k(\Delta_t A_1, \Delta_t A_2, \ldots, \Delta_t A_k) 1_{\{N_s^{\Delta} = k\}}] = \mathbb{E}[\Phi_k(\lambda_1 e_1, \lambda_2 e_2, \ldots, \lambda_k e_k) 1_{\{N_s^{c_0,c_1} = k\}}]$, *where λ_k has been defined by* (2.2).

Proof. 1) Statement 1) follows from:

$$\begin{aligned} \mathbb{P}(N_s^{\Delta} = 0) &= \mathbb{P}(N_{\lfloor s/\Delta_t \rfloor} = 0) = \mathbb{P}(T_1 \geq \lfloor s/\Delta_t \rfloor) \\ &= \mathbb{P}(A_1 \geq \lfloor s/\Delta_t \rfloor) = \mathbb{P}(\Delta_t A_1 \geq \Delta_t \lfloor s/\Delta_t \rfloor) \end{aligned}$$

where $\lfloor a \rfloor$ denotes the integer part of a.
2) Set $k \geq 1$. The event $\{N_s^\Delta = k\}$ can be decomposed as follows:

$$\{N_s^\Delta = k\} = \left\{\Delta_t \sum_{j=1}^{k} A_j \leq s\right\} \cap \left\{\Delta_t \sum_{j=1}^{k+1} A_j > s\right\}.$$

This identity imply existence of a bounded Borel function $\psi_k : \mathbb{R}^{k+1} \to \mathbb{R}$ so that

$$\begin{aligned}
&\Phi_k(\Delta_t A_1, \dots, \Delta_t A_k) 1_{\{N_s^\Delta = k\}} \\
&= \Phi_k(\Delta_t A_1, \dots, \Delta_t A_k) 1_{\{\Delta_t \sum_{j=1}^{k} A_j \leq s\}} 1_{\{\Delta_t \sum_{j=1}^{k+1} A_j > s\}} \\
&= \psi_k(\Delta_t A_1, \Delta_t A_2, \dots, \Delta_t A_{k+1}).
\end{aligned}$$

Since Φ_k is continuous, the discontinuity points of ψ_k are included in:

$$\mathbb{U} = \left\{x \in \mathbb{R}^{k+1} :\ \sum_{j=1}^{k} x_j = s\right\} \cup \left\{x \in \mathbb{R}^{k+1} :\ \sum_{j=1}^{k+1} x_j = s\right\}.$$

By Lemma 4.1, $(\Delta_t A_1, \dots, \Delta_t A_{k+1})$ converges in distribution towards $(\lambda_1 e_1, \dots, \lambda_{k+1} e_{k+1})$ as $\Delta_t \to 0$. Since the Lebesgue measure of $\mathbb{U}$ is null, the limit law does not charge $\mathbb{U}$. We can conclude evoking for instance Theorem 14 p.247 in [3]). □

Let us formulate a straightforward generalization of Lemma 4.2.

Lemma 4.3. *Let $n \in \mathbb{N}$, $(k_1, \dots, k_n) \in \mathbb{N}^n$ such that $k_1 \leq k_2 \leq \dots \leq k_n$ and $(s_1, \dots, s_n) \in \mathbb{R}_+^n$ with $s_1 \leq s_2 \leq \dots \leq s_n$. Let $\Phi : \mathbb{R}^{k_n} \to \mathbb{R}$ be a bounded and continuous function. Then*

$$\begin{aligned}
&\lim_{\Delta_t \to 0} \mathbb{E}[\Phi(\Delta_t A_1, ..., \Delta_t A_{k_n}) 1_{\{N_{s_1}^\Delta = k_1, \dots, N_{s_n}^\Delta = k_n\}}] \\
&\quad = \mathbb{E}[\Phi(\lambda_1 e_1, ..., \lambda_{k_n} e_{k_n}) 1_{\{N_{s_1}^{c_0,c_1} = k_1, \dots, N_{s_n}^{c_0,c_1} = k_n\}}] \qquad (4.10)
\end{aligned}$$

Proposition 4.4. *The random variable $\tilde{Z}_s^\Delta$ converges in distribution towards $-Z_s^{c_0,c_1}$, for any $s > 0$, as $\Delta_t \to 0$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which is bounded by M. Identities (4.8) and (4.5) imply that $\mathbb{E}[f(\tilde{Z}_s^\Delta)] = \sum_{k=0}^{\infty} E_\Delta(k)$, with

$$E_\Delta(k) = \mathbb{E}\left[f\Big(\sum_{j=1}^{k} (-1)^j \Delta_t A_j + (-1)^{k+1}\Big(s - \Delta_t \sum_{j=1}^{k} A_j + \Delta_t\Big)\Big) 1_{\{N_s^\Delta = k\}}\right]$$

Applying Lemma 4.2 and (3.8), we obtain for any $k \geq 0$,

$$\begin{aligned}
\lim_{\Delta_t \to 0} E_\Delta(k) &= E\Big[f\Big(\sum_{j=1}^{k} (-1)^j \lambda_j e_j + (-1)^{k+1}\Big(s - \sum_{j=1}^{k} \lambda_j e_j\Big)\Big) 1_{\{N_s^{c_0,c_1} = k\}}\Big] \\
&= \mathbb{E}[f(-Z_s^{c_0,c_1}) 1_{\{N_s^{c_0,c_1} = k\}}].
\end{aligned}$$

Moreover since f is bounded by M, we get

$$|E_\Delta(k)| \leq M\, \mathbb{P}(N_s^\Delta = k).$$

Suppose that $k \geq 1$. Then, using the Markov inequality and the independence property of the random sequence $(A_n,\ n \geq 0)$, we obtain

$$\begin{aligned}
\mathbb{P}(N_s^\Delta = k) &= \mathbb{P}\left(\Delta_t \sum_{j=1}^{k} A_j \leq s < \Delta_t \sum_{j=1}^{k+1} A_j\right) \\
&\leq \mathbb{P}\left(\Delta_t \sum_{j=1}^{k} A_j \leq s\right) = \mathbb{P}\left(\exp\Big\{-\Delta_t \sum_{j=1}^{k} A_j\Big\} \geq e^{-s}\right) \\
&\leq e^s\, \mathbb{E}\left[\exp -\Delta_t \sum_{j=1}^{k} A_j\right] = e^s \prod_{j=1}^{k} \varphi_j(1)
\end{aligned}$$

where $\varphi_j(\mu) = \mathbb{E}[e^{-\mu\Delta_t A_j}]$. Since (Y_n) is a Markov chain starting at $Y_0 = -1$, for any $j \geq 1$, A_{2j-1} (resp. A_{2j}) is geometrically distributed with parameter $c_0\Delta_t$ (resp. $c_1\Delta_t$). According to (4.9) we get

$$\varphi_{2j}(1) = \frac{c_1\Delta_t}{e^{\Delta_t} - 1 + c_1\Delta_t} \leq \frac{c_1\Delta_t}{\Delta_t + c_1\Delta_t} = \frac{c_1}{1+c_1} < 1.$$

By the same way, we have:

$$\varphi_{2j-1}(1) \leq \frac{c_0}{1+c_0} < 1.$$

As a result, there exists $0 < r < 1$ so that

$$\mathbb{P}(N_s^\Delta = k) \leq e^s r^k. \tag{4.11}$$

We are now allowed to apply the dominated convergence theorem:

$$\lim_{\Delta_t \to 0} \mathbb{E}[f(\tilde{Z}_s^\Delta)] = \sum_{k \geq 0} \lim_{\Delta_t \to 0} E_\Delta(k) = \sum_{k \geq 0} \mathbb{E}[f(-Z_s^{c_0,c_1}) 1_{\{N_s^{c_0,c_1} = k\}}] = \mathbb{E}[f(-Z_s^{c_0,c_1})].$$

□

Proposition 4.5. *For any $(s_1, \ldots, s_n) \in \mathbb{R}_+^n$ such that $s_1 \leq s_2 \leq \ldots \leq s_n$, the random vector $(\tilde{Z}_{s_1}^\Delta, \ldots, \tilde{Z}_{s_n}^\Delta)$ converges in distribution to $(-Z_{s_1}^{c_0,c_1}, \ldots, -Z_{s_n}^{c_0,c_1})$, as Δ_t tends to 0.*

Proof. We follow the approach developed in the proof of Proposition 4.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded and continuous function. We have:

$$\mathbb{E}\left[f(\tilde{Z}_{s_1}^\Delta, \ldots, \tilde{Z}_{s_n}^\Delta)\right] = \sum_{k_1, \ldots, k_n} E_\Delta(k_1, \ldots, k_n),$$

where the sum is extended to $(k_1, \ldots, k_n) \in \mathbb{N}^n$ so that $k_1 \leq k_2 \leq \ldots \leq k_n$ and

$$E_\Delta(k_1, \ldots, k_n) = \mathbb{E}\left[f(\tilde{Z}_{s_1}^\Delta, \ldots, \tilde{Z}_{s_n}^\Delta) 1_{\{N_{s_1}^\Delta = k_1, \ldots, N_{s_n}^\Delta = k_n\}}\right].$$

Identity (4.8) implies the existence of a bounded continuous function $\psi_n : \mathbb{R}^{k_n} \to \mathbb{R}$ so that

$$E_\Delta(k_1, \ldots, k_n) = \mathbb{E}\left[\psi_n(\Delta_t A_1, \ldots, \Delta_t A_{k_n}) 1_{\{N_{s_1}^\Delta = k_1, \ldots, N_{s_n}^\Delta = k_n\}}\right].$$

Applying Lemma 4.3, we get

$$\lim_{\Delta_t \to 0} E_\Delta(k_1, \ldots, k_n) = \mathbb{E}[\psi_n(\lambda_1 e_1, \ldots, \lambda_{k_n} e_{k_n}) 1_{\{N_{s_1}^{c_0,c_1} = k_1, \ldots, N_{s_n}^{c_0,c_1} = k_n\}}].$$

According to the definition of the process $Z_s^{c_0,c_1}$, we may deduce :

$$\lim_{\Delta_t \to 0} E_\Delta(k_1, \ldots, k_n) = \mathbb{E}[f(-Z_{s_1}^{c_0,c_1}, \ldots, -Z_{s_n}^{c_0,c_1}) 1_{\{N_{s_1}^{c_0,c_1} = k_1, \ldots, N_{s_n}^{c_0,c_1} = k_n\}}].$$

In order to obtain that

$$\lim_{\Delta_t \to 0} \mathbb{E}[f(\tilde{Z}_{s_1}^\Delta, \ldots, \tilde{Z}_{s_n}^\Delta)] = \mathbb{E}[f(-Z_{s_1}^{c_0,c_1}, \ldots, -Z_{s_n}^{c_0,c_1})],$$

it suffices (cf the proof of Proposition 4.4) to prove that

$$\sum_{k_1, \ldots, k_{n-1}} \sup_{\Delta_t} |E_\Delta(k_1, \ldots, k_n)| < \infty.$$

Since f is bounded,

$$|E_\Delta(k_1, \ldots, k_n)| \leq M\, \mathbb{P}(N_{s_n}^\Delta = k_n)$$

Using moreover (4.11) we get

$$\sum_{k_1, \ldots, k_n} |E_\Delta(k_1, \ldots, k_n)| \leq M e^{s_n} \sum_{k_n} (k_n)^{n-1} r^{k_n} < \infty$$

since $r < 1$.

□

We are now able to complete the proof of Theorem 2.1. Since $(\tilde{Z}_s^\Delta)$ and $(Z_s^{c_0,c_1})$ are both continuous processes, the convergence of the process $(\tilde{Z}_s^\Delta)$ to the process $(-Z_s^{c_0,c_1})$ will be proved as soon as the following measure tension criterium (cf Theorem 8.3 p.56 in [2]) holds : for all $\varepsilon > 0$ and η_0, there exists some constants $\delta \in]0,1[$ and $\mu > 0$ such that

$$\frac{1}{\delta}\, \mathbb{P}\Big(\sup_{s \le u \le s+\delta} |\tilde{Z}_u^\Delta - \tilde{Z}_s^\Delta| \ge \varepsilon \Big) \le \eta_0, \quad \text{for any } \Delta_t \le \mu. \tag{4.12}$$

Since $(\tilde{Z}_s^\Delta,\ s \ge 0)$ is the interpolated persistent random walk, its slope is always equal to 1 or -1. Hence we obtain for any $(u,s) \in \mathbb{R}_+^2$,

$$|\tilde{Z}_u^\Delta - \tilde{Z}_s^\Delta| \le |u-s|.$$

Consequently

$$\sup_{s \le u \le s+\delta} |\tilde{Z}_u^\Delta - \tilde{Z}_s^\Delta| \le \delta.$$

By choosing $\delta = \varepsilon/2$ we get the tension criterium and so the convergence of the process $(\tilde{Z}_s^\Delta)$ to the process $(-Z_s^{c_0,c_1})$.

5 Two extensions of Theorem 2.1

First of all, the extensions presented in this section concerns the regime $\Delta_x = \Delta_t$.

5.1 The case when (Y_t) takes k values.

Let us introduce our parameters. Let $k \ge 2$, $y_1, \dots, y_k$ denote k real numbers, and $(c(i,j);\ 1 \le i,j \le k)$ a matrix so that

$$c(i,j) \ge 0 \text{ for any } i \ne j,\ c(i,i) = 0,\ \sum_{l=1}^{k} c(i,l) > 0\ \forall i. \tag{5.1}$$

We directly consider the asymptotic regime. Let (Y_t) be a $\{y_1, \dots, y_k\}$-valued Markov chain, with transition probability matrix:

$$\pi^\Delta(y_i, y_j) = \begin{cases} c(i,j)\Delta_t & i \ne j \\ 1 - \Big(\sum_{l=1}^{k} c(i,l)\Big)\Delta_t & i = j, \end{cases} \tag{5.2}$$

where $\Delta_t > 0$ is supposed to be small so that

$$c(i,j)\Delta_t \le 1, \quad \Big(\sum_{l=1}^{k} c(i,l)\Big)\Delta_t < 1.$$

Similarly to the case $k = 2$ and $y_1 = -1$, $y_2 = 1$, we are interested in the linear interpolation $(\tilde{Z}_s^\Delta;\ s \ge 0)$ of the process $(Z_s^\Delta;\ s \ge 0)$ defined by (1.7).

Theorem 5.1. *Suppose $Y_0 = y_i$. Then $(\tilde{Z}_s^\Delta;\ s \ge 0)$ converges in distribution, as $\Delta_t \to 0$, to the process $\Big(\int_0^t R_s ds;\ t \ge 0\Big)$ where (R_s) is a $\{y_1, \dots, y_k\}$-valued continuous-time Markov chain starting at level y_i, whose dynamic is the following: (R_t) stays on level y_i an exponential time with parameter $1/\Big(\sum_{l=1}^{k} c(j,l)\Big)$ and jumps to $y_{j'}$ $(j' \ne j)$ with probability $c(j,j')/\Big(\sum_{l=1}^{k} c(j,l)\Big)$.*

Remark 5.2. *In the case $k = 2$, $y_1 = -1$ and $y_2 = 1$, then $((-1)^{N_t^{c_0,c_1}};\ t \ge 0)$ (cf (2.1)) may be chosen as a realization of (R_t) when it starts at $R_0 = -1$.*

Proof of Theorem 5.1. We proceed as in the proof of Theorem 2.1 developed in Section 4. Let $(T_n)_{n \ge 1}$ be the sequence of stopping times defined by (4.1). Then:

$$X_t = \begin{cases} y_i(t+1) & 0 \le t < T_1 \\ y_i T_1 + Y_{T_1}(t - T_1 + 1) & T_1 \le t < T_2. \end{cases}$$

Recall that $(Z_s^\Delta;\ s/\Delta_t \in \mathbb{N})$ has been defined by (4.5). From the relations above, it is easy to deduce:

$$Z_s^\Delta = \begin{cases} y_i(s+\Delta_t) & 0 \le s \le \Delta_t T_1 \\ y_i(\Delta_t T_1) + Y_{T_1}(s - \Delta_t T_1 + \Delta_t) & \Delta_t T_1 \le s < \Delta_t T_2. \end{cases}$$

Let us determine the limit distribution of $(\Delta_t T_1, Y_{T_1})$ as $\Delta_t \to 0$. Set

$$V^\Delta(\lambda, j) = \mathbb{E}\left[e^{-\lambda \Delta_t T_1} 1_{\{Y_{T_1} = y_j\}}\right], \quad \lambda > 0,\ j \neq i.$$

Proceeding as in the proof of Lemma 4.1, we obtain:

$$V^\Delta(\lambda, j) = \frac{e^{-\lambda\Delta_t} c(i,j)\Delta_t}{1 - \left[1 - \left(\sum_{l=1}^k c(i,l)\right)\Delta_t\right] e^{-\lambda\Delta_t}}.$$

Using standard analysis, we deduce that $(\Delta_t T_1, Y_{T_1})$ converges in distribution as $\Delta_t \to 0$ to (e_1', U_1) where:

$$\mathbb{E}\left[e^{-\lambda e_1'} 1_{\{U_1 = j\}}\right] = \frac{c(i,j)}{\lambda + \sum_{l=1}^k c(i,l)}.$$

As a result, e_1' and U_1 are independent, e_1' is exponentially distributed with parameter $1/\sum_{l=1}^k c(i,l)$ and

$$\mathbb{P}(U_1 = j) = \frac{c(i,j)}{\sum_{l=1}^k c(i,l)}.$$

Using the approach developed in Section 4, we can prove Theorem 5.1. The details are left to the reader. □

5.2 The case when (Y_t) is a Markov chain of order 2.

Let (Y_t) be a Markov chain with order 2. For simplicity we suppose that it takes its values in $\{-1, 1\}$. Obviously $(Y_t, Y_{t+1})_{t\ge 0}$ is a Markov chain with state space

$$E = \{(-1,-1), (-1,1), (1,-1), (1,1)\}.$$

Let π^Δ be the transition probability matrix:

$$\pi^\Delta = \begin{pmatrix} 1 - c_0\Delta_t & c_0\Delta_t & 0 & 0 \\ 0 & 0 & 1-p_0 & p_0 \\ p_1 & 1-p_1 & 0 & 0 \\ 0 & 0 & c_1\Delta_t & 1 - c_1\Delta_t \end{pmatrix} \tag{5.3}$$

where $\Delta_t, c_0, c_1, p_0, p_1 > 0$ and $c_0\Delta_t, c_1\Delta_t, p_0, p_1 < 1$.
Let us introduce:

$$v_i = \frac{p_i}{1 - (1-p_0)(1-p_1)}, \quad c_i' = c_i v_i,\ i = 0, 1. \tag{5.4}$$

Recall that (Z_t^Δ) and $(\tilde Z_t^\Delta)$ have been defined by (4.6), resp. (4.8), $(N_t^{c_0,c_1})$ is the counting process defined by (2.1), and

$$Z_t^{c_0,c_1} = \int_0^t (-1)^{N_u^{c_0,c_1}} du,\ t \ge 0.$$

Theorem 5.3. *1) Suppose that $Y_0 = Y_1 = -1$ (resp. $Y_0 = Y_1 = 1$) then $(\tilde Z_s^\Delta;\ s \ge 0)$ converges in distribution, as $\Delta_t \to 0$, to $(-Z_s^{c_0',c_1'};\ s \ge 0)$ (resp. $(Z_s^{c_1',c_0'};\ s \ge 0)$).*
2) Suppose $Y_0 = 1$ and $Y_1 = -1$ (resp. $Y_0 = -1$, $Y_1 = 1$) then $(\tilde Z_s^\Delta;\ s \ge 0)$ converges in distribution, as $\Delta_t \to 0$, to

$$\left((\epsilon - 1)\int_0^s (-1)^{N_u^{c_0',c_1'}} du + \epsilon \int_0^s (-1)^{N_u^{c_1',c_0'}} du;\ s \ge 0\right)$$

where ϵ is independent from $(N_u^{c_0',c_1'})$, $(N_u^{c_1',c_0'})$ and

$$\mathbb{P}(\epsilon = 0) = 1 - \mathbb{P}(\epsilon = 1) = v_1 \quad (\textit{resp. } \mathbb{P}(\epsilon = 1) = 1 - \mathbb{P}(\epsilon = 0) = v_0).$$

Remark 5.4. *1) Note that $(Y_t)_{t\in\mathbb{N}}$ is a Markov chain if and only if $1-c_0\Delta_t = p_1$ and $1-c_1\Delta_t = p_0$. If we replace formally p_0 (resp. p_1) by $1-c_1\Delta_t$ (resp. $1-c_0\Delta_t$) in (5.4) and take the limit $\Delta_t \to 0$, we obtain $v_i = p_i$ and $c'_i = c_i$. We recover Theorem 2.1.*
2) The fact that (Y_t) is a Markov chain with order 2 does not modify drastically the limit. The limit process can be expressed in terms of processes of the type $(Z_s^{\alpha,\beta};\ s\geq 0)$.

Proof of Theorem 5.3. **1)** We only consider the case $Y_0 = Y_1 = 1$. Let us define T_1, T_2 and T_3 as follows:

$$T_1 = \inf\{t\geq 1,\ Y_t=-1\},\quad T_2=\inf\{t\geq T_1+1,\ Y_t=Y_{t-1}\},\quad T_3=\inf\{t\geq T_2+1,\ Y_t\neq Y_{T_2}\}.$$

Using the definition (cf (1.1)) of (X_t) we easely obtain:

$$X_t = \begin{cases} t+1 & 0\leq t<T_1 \\ T_1+\hat{X}_t & T_1\leq t<T_2 \end{cases}$$

where $\hat{X}_t$ equals either -1 or 0.
Moreover, when $T_2\leq t<T_3$, we have:

$$X_t = \begin{cases} T_1-2-(t-T_2) & \text{if } T_2-T_1 \text{ is odd} \\ T_1+1+(t-T_2) & \text{otherwise.} \end{cases}$$

According to (1.7), we can deduce:

$$Z_s^{\Delta} = \begin{cases} s+\Delta_t & 0\leq s\leq \Delta_t T_1 \\ \Delta_t T_1+\Delta_t\hat{X}_{s/\Delta_t} & \Delta_t T_1\leq s<\Delta_t T_2 \\ \Delta_t T_1-2\Delta_t-(s-\Delta_t T_2) & \Delta_t T_2\leq s<\Delta_t T_3,\ Y_{T_2}=-1 \\ \Delta_t T_1+\Delta_t+s-\Delta_t T_2 & \Delta_t T_2\leq s<\Delta_t T_3,\ Y_{T_2}=1 \end{cases}$$

(note that T_2-T_1 is odd if and only if $Y_{T_2}=-1$).
2) a) Proceeding as in the proof of Theorem 2.1, we can prove that $\Delta_t T_1$ converges in distribution, as $\Delta_t\to 0$, to e'_1, where e'_1 is exponentially distributed with parameter $1/c_1$. Then $(\tilde{Z}_s^{\Delta};\ 0\leq s\leq \Delta_t T_1)\xrightarrow{(d)}(s;\ s\leq e'_1)$, as $\Delta_t\to 0$.
b) The distribution of T_2-T_1 does not depend on Δ_t. Moreover $|\hat{X}_\cdot|\leq 1$, then the limit of the length of the interval $[\Delta_t T_1,\Delta_t T_2]$ is null. We have

$$\mathbb{P}(Y_{T_2}=-1)=\sum_{l\geq 0}\Big((1-p_1)(1-p_0)\Big)^l p_1 = v_1.$$

c) Using the strong Markov property, we easely show that $(\tilde{Z}^{\Delta}_{s+\Delta_t T_2};\ 0\leq s\leq \Delta_t(T_3-T_1))\xrightarrow{(d)}(e'_1+Y_{T_1}s;\ 0\leq s\leq e'_2)$, as $\Delta_t\to 0$, where (e'_1,Y_{T_1}) (resp. (e'_1,e'_2)) are independent r.v.'s and conditionally on $Y_{T_2}=1$ (resp. $Y_{T_2}=-1$) e'_2 is exponentially distributed with parameter $1/c_1$ (resp. $1/c_0$).
d) Let us summarize the former analysis. We have proved that $(\tilde{Z}_s^{\Delta};\ s\geq 0)\xrightarrow{(d)}(\int_0^s\hat{R}_u du,\ s\geq 0)$, where $(\hat{R}_u)$ is a continuous-time Markov chain which takes its values in $\{-1,1\}$ and $\hat{R}_0=1$. Moreover the dynamic of $(\hat{R}_u)$ is the following: $(\hat{R}_u)$ stays in 1 (resp. -1) an exponential time with parameter $1/c_1$ (resp. $1/c_0$) and moves to -1 (resp. 1) with probability v_1 (resp. v_0). Note that $(\hat{R}_u)$ is allowed to stay in the same site. It is classical (cf [12]) to prove that $(\hat{R}_u)_{u\geq 0}\overset{(d)}{=}(Z_u^{c'_1,c'_0})_{u\geq 0}$ where c'_0 and c'_1 are defined by (5.4). □

6 Convergence of the persistent random walk towards the Brownian motion with drift

In subsection 6.1 below we determine the generating function of X_t, where X_t is the persistent random walk defined by (1.1). This allows to prove Theorem 2.2 and Proposition 2.3 in subsections 6.2, 6.3.

6.1 The moment generating function of X_t

Let us recall that the increments process $(Y_t,\, t \in \mathbb{N})$ is a Markov chain valued in the state space $E = \{-1, 1\}$. Its transition probability is given by

$$\pi = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \qquad 0 < \alpha < 1, \quad 0 < \beta < 1.$$

The persistent random walk $(X_t,\, t \in \mathbb{N})$ is defined by the partial sum:

$$X_t = \sum_{i=0}^{t} Y_i \quad \text{with} \quad X_0 = Y_0 = 1 \text{ or } -1.$$

Lemma 6.1. *Let us define the functions a_t and b_t:*

$$a_t(j) = \mathbb{P}(X_t = j, Y_t = -1) \quad \textit{and} \quad b_t(j) = \mathbb{P}(X_t = j, Y_t = 1). \tag{6.1}$$

Then,

$$a_{t+1}(j) = (1-\alpha)a_t(j+1) + \beta b_t(j+1) \tag{6.2}$$
$$b_{t+1}(j) = \alpha a_t(j-1) + (1-\beta)b_t(j-1). \tag{6.3}$$

Proof. Using the Markov property of (Y_t) we have:

$$\begin{aligned} a_{t+1}(j) &= \mathbb{P}(X_{t+1} = j, Y_{t+1} = -1, Y_t = -1) + \mathbb{P}(X_{t+1} = j, Y_{t+1} = -1, Y_t = 1) \\ &= \mathbb{P}(X_t = j+1, Y_{t+1} = -1, Y_t = -1) + \mathbb{P}(X_t = j+1, Y_{t+1} = -1, Y_t = 1) \\ &= (1-\alpha)a_t(j+1) + \beta b_t(j+1). \end{aligned}$$

The second recursive formula involving $(b_t(j))$ can be obtained similarly. □

Let us define the moment generating function $\Phi(\lambda, t) = \mathbb{E}[\lambda^{X_t}], \quad (\lambda > 0)$. We decompose $\Phi(\lambda, t)$ as

$$\Phi(\lambda, t) = \Phi_-(\lambda, t) + \Phi_+(\lambda, t), \tag{6.4}$$

with

$$\Phi_-(\lambda, t) = \mathbb{E}[\lambda^{X_t} 1_{\{Y_t = -1\}}], \qquad \Phi_+(\lambda, t) = \mathbb{E}[\lambda^{X_t} 1_{\{Y_t = 1\}}]. \tag{6.5}$$

Lemma 6.2. *1) $\Phi_-(\lambda, 0) = \frac{1}{\lambda}\,\mathbb{P}(Y_0 = -1)$ and $\Phi_+(\lambda, 0) = \lambda\,\mathbb{P}(Y_0 = 1)$.*
2) The moment generating function verifies the following induction equations:

$$\Phi_-(\lambda, t+1) = \frac{1-\alpha}{\lambda}\,\Phi_-(\lambda, t) + \frac{\beta}{\lambda}\,\Phi_+(\lambda, t) \tag{6.6}$$
$$\Phi_+(\lambda, t+1) = \alpha\lambda\Phi_-(\lambda, t) + (1-\beta)\lambda\Phi_+(\lambda, t) \tag{6.7}$$

Proof. Definition (6.1) implies that

$$\Phi_-(\lambda, t) = \sum_{j \in \mathbb{Z}} \lambda^j a_t(j) = \sum_{j=-t-1}^{t+1} \lambda^j a_t(j).$$

Hence,

$$\begin{aligned} \Phi_-(\lambda, t+1) &= \sum_j \lambda^j a_{t+1}(j) = (1-\alpha)\sum_j \lambda^j a_t(j+1) + \beta \sum_j \lambda^j b_t(j+1) \\ &= (1-\alpha)\frac{1}{\lambda}\sum_j \lambda^{j+1} a_t(j+1) + \frac{\beta}{\lambda}\sum_j \lambda^{j+1} b_t(j+1) \\ &= \frac{1-\alpha}{\lambda}\Phi_-(\lambda, t) + \frac{\beta}{\lambda}\Phi_+(\lambda, t). \end{aligned}$$

The proof of (6.7) is similar. □

Lemma 6.3. *Let $f(\lambda, t)$ be equal to either $\Phi_-(\lambda, t)$ or $\Phi_+(\lambda, t)$, then*

$$f(\lambda, t+2) - \Big(\frac{1-\alpha}{\lambda} + (1-\beta)\lambda\Big) f(\lambda, t+1) + (1-\alpha-\beta) f(\lambda, t) = 0. \tag{6.8}$$

Proof. By (6.6), we get

$$\Phi_+(\lambda,t)=\Big\{\Phi_-(\lambda,t+1)-\frac{1-\alpha}{\lambda}\Phi_-(\lambda,t)\Big\}\frac{\lambda}{\beta}. \tag{6.9}$$

Replacing t by $t+1$ in (6.9), we obtain

$$\Phi_+(\lambda,t+1)=\Big\{\Phi_-(\lambda,t+2)-\frac{1-\alpha}{\lambda}\Phi_-(\lambda,t+1)\Big\}\frac{\lambda}{\beta}. \tag{6.10}$$

Using successively (6.7), (6.10) and (6.9), we have:

$$\begin{aligned}\alpha\lambda\Phi_-(\lambda,t) &= \Phi_+(\lambda,t+1)-(1-\beta)\lambda\Phi_+(\lambda,t)\\ &= \frac{\lambda}{\beta}\left\{\Phi_-(\lambda,t+2)-\frac{1-\alpha}{\lambda}\Phi_-(\lambda,t+1)\right\}\\ &\quad -\frac{1-\beta}{\beta}\lambda^2\left\{\Phi_-(\lambda,t+1)-\frac{1-\alpha}{\lambda}\Phi_-(\lambda,t)\right\}.\end{aligned}$$

Finally

$$\Phi_-(\lambda,t+2)-\left(\frac{1-\alpha}{\lambda}+(1-\beta)\lambda\right)\Phi_-(\lambda,t+1)+\Big((1-\alpha)(1-\beta)-\alpha\beta\Big)\Phi_-(\lambda,t)=0.$$

The proof concerning $f(\lambda,t)=\Phi_-(\lambda,t)$ is similar and is left to the reader. □

In order to obtain the explicit form of $\Phi_-(\lambda,t)$ and $\Phi_+(\lambda,t)$ in terms of λ and t, it suffices to compute the roots ϑ_- and ϑ_+ of the following polynomial

$$\vartheta^2-\left(\frac{1-\alpha}{\lambda}+(1-\beta)\lambda\right)\vartheta+1-\alpha-\beta=0 \tag{6.11}$$

Its discriminant equals

$$\mathcal{D}=\Big(\frac{1-\alpha}{\lambda}+(1-\beta)\lambda\Big)^2-4(1-\alpha-\beta). \tag{6.12}$$

It is clear that

$$\begin{aligned}\mathcal{D} &= \Big(\frac{1-\alpha}{\lambda}+(1-\beta)\lambda+2\sqrt{\rho}\Big)\Big(\frac{1-\alpha}{\lambda}+(1-\beta)\lambda-2\sqrt{\rho}\Big)\\ &= \frac{1}{\lambda}\Big(\frac{1-\alpha}{\lambda}+(1-\beta)\lambda+2\sqrt{\rho}\Big)\Big((1-\beta)\lambda^2-2\sqrt{\rho}\lambda+1-\alpha\Big).\end{aligned} \tag{6.13}$$

Since the discriminant of $\lambda\to(1-\beta)\lambda^2-2\sqrt{\rho}\lambda+1-\alpha$ is equal to $-4\alpha\beta$ then $\mathcal{D}>0$ for any $\lambda>0$.
Consequently the roots of (6.11) are:

$$\vartheta_\pm=\frac{1}{2}\Big(\frac{1-\alpha}{\lambda}+(1-\beta)\lambda\pm\sqrt{\mathcal{D}}\Big). \tag{6.14}$$

We deduce the following result.

Proposition 6.4. *1) The moment generating function $\Phi(\lambda,t)$ satisfies*

$$\Phi(\lambda,t)=a_+\vartheta_+^t+a_-\vartheta_-^t \tag{6.15}$$

with

$$a_+=\frac{1-\alpha+\lambda(\lambda\alpha-\vartheta_-)}{\lambda^2\sqrt{\mathcal{D}}}\quad \textit{and}\quad a_-=\frac{1}{\lambda}-a_+\quad \textit{if } X_0=Y_0=-1$$

and

$$a_+=\frac{(1-\beta)\lambda^2+\beta-\lambda\vartheta_-}{\sqrt{\mathcal{D}}}\quad \textit{and}\quad a_-=\lambda-a_+\quad \textit{if } X_0=Y_0=1.$$

Proof. Suppose that $X_0 = Y_0 = -1$. Let us first determine the values of the generating function at time $t = 0$ and $t = 1$:

$$\Phi(\lambda, 0) = \Phi_+(\lambda, 0) + \Phi_-(\lambda, 0) = \frac{1}{\lambda}\,\mathbb{P}(Y_0 = -1) + \lambda\,\mathbb{P}(Y_0 = 1) = \frac{1}{\lambda} = a_+ + a_-$$

Moreover, using (6.6) and (6.7) with $t = 0$, we get

$$\Phi(\lambda, 1) = \Phi_+(\lambda, 1) + \Phi_-(\lambda, 1) = \Big(\frac{1-\alpha}{\lambda} + \alpha\lambda\Big)\Phi_-(\lambda, 0) = \frac{1-\alpha}{\lambda^2} + \alpha = a_+\vartheta_+ + a_-\vartheta_-.$$

It is clear that Lemma 6.3 and $\Phi(\lambda, t) = \Phi_+(\lambda, t) + \Phi_-(\lambda, t)$ implies that $\Phi(\lambda, t)$ satisfies (6.8). Then (6.15) follows by standard arguments. The second case $X_0 = Y_0 = 1$ can be proved in a similar way. □

6.2 Proof of Theorem 2.2

We keep the notations given in Section 1. Let α_0 and β_0 be two real numbers in $[0, 1]$. Let Δ_x be a small space parameter so that:

$$0 \le \alpha_0 + c_0\Delta_x \le 1, \quad 0 \le \beta_0 + c_1\Delta_x \le 1,$$

where c_0 and c_1 belong to $\mathbb{R}$.
Note that $\alpha_0 > 0$ (resp. $\beta_0 > 0$) implies that $\alpha_0 + c_0\Delta_x > 0$ (resp. $\beta_0 + c_1\Delta_x > 0$) when Δ_x is small enough. If $\alpha_0 < 1$ (resp. $\beta_0 < 1$), similarly $\alpha_0 + c_0\Delta_x < 1$ (resp. $\beta_0 + c_1\Delta_x < 1$) as soon as Δ_x is small. In the case $\alpha_0 = 1$ (resp. $\beta_0 = 1$) c_0 (resp. c_1) has to be chosen in $]-\infty, 0]$.
We assume that the coefficients of the transition probability matrix π^Δ of the Markov chain (Y_t) satisfy:

$$\alpha = \alpha_0 + c_0\Delta_x, \quad \beta = \beta_0 + c_1\Delta_x \tag{6.16}$$

i.e. π^Δ is given by (1.6). (X_t) is defined by (1.1) and (Z_s^Δ) is the normalized persistent random walk:

$$Z_s^\Delta = \Delta_x X_{s/\Delta_t}, \quad (\Delta_t > 0,\ \Delta_x > 0,\ s \in \Delta_t\,\mathbb{N}).$$

$(\tilde{Z}_t^\Delta;\ t \ge 0)$ denotes the linear interpolation of (Z_t^Δ).
Recall that $\rho_0 = 1 - \alpha_0 - \beta_0$ and $\eta_0 = \beta_0 - \alpha_0$. Note that $\rho_0 \ne 1 \Longleftrightarrow \alpha_0 + \beta_0 \ne 0$

Proposition 6.5. *Let $\rho_0 \ne 1$,*
1) if $r\Delta_t = \Delta_x$ with $r > 0$ then $\tilde{Z}_t^\Delta$ converges towards the deterministic limit $-\frac{rt\eta_0}{1-\rho_0}$ as Δ_x tends to 0.
2) if $r\Delta_t = \Delta_x^2$ with $r > 0$, $\tilde{Z}_t^\Delta + \frac{t\sqrt{r}\eta_0}{(1-\rho_0)\sqrt{\Delta_t}}$ converges in distribution to the Gaussian law with mean

$$m = rt\Big(\frac{-\overline{c}}{1-\rho_0} + \frac{\eta_0 c}{(1-\rho_0)^2}\Big) \tag{6.17}$$

and variance

$$\sigma^2 = \frac{r(1+\rho_0)}{1-\rho_0}\Big(1 - \frac{\eta_0^2}{(1-\rho_0)^2}\Big)t, \tag{6.18}$$

where

$$c = c_0 + c_1 \quad \textit{and} \quad \overline{c} = c_1 - c_0. \tag{6.19}$$

Proof. We shall prove the statement under the condition $X_0 = Y_0 = -1$. If $X_0 = Y_0 = +1$, the limit is obtained by changing the sign and replacing c_0 (resp. c_1) by c_1 (resp. c_0).
1) Let $\Phi(\lambda, t)$ be the generating function associated with X_t. In order to determine the limit distribution of Z_t^Δ, let us introduce:

$$\phi(\mu, t) = \mathbb{E}_{-1}[e^{-\mu\tilde{Z}_t^\Delta}], \tag{6.20}$$

where $\mathbb{E}_{-1}$ denotes the expectation when $Y_0 = -1$. Observe that

$$\phi(\mu, t) = \Phi(e^{-\mu\Delta_x}, \frac{t}{\Delta_t}) = \mathbb{E}_{-1}[e^{-\mu\Delta_x X(t/\Delta_t)}], \tag{6.21}$$

when $t/\Delta_t \in \mathbb{N}$.
According to Proposition 6.4, when $t/\Delta_t \in \mathbb{N}$, $\phi(\mu, t)$ can be expressed in terms of a_+, a_-

and $\sqrt{\mathcal{D}}$.
First let us study the asymptotic expansion of the discriminant $\mathcal{D}$ as $\Delta_x \to 0$. It is convenient to set:

$$\bar{\delta} = c_0\Delta_x \quad \text{and} \quad \hat{\delta} = c_1\Delta_x. \tag{6.22}$$

Applying (6.12) with $\alpha = \alpha_0 + \bar{\delta}$ and $\beta = \beta_0 + \hat{\delta}$ we have:

$$\mathcal{D} = \Big((1-\alpha_0-\bar{\delta})e^{\mu\Delta_x} + (1-\beta_0-\hat{\delta})e^{-\mu\Delta_x}\Big)^2 - 4(1-\alpha_0-\beta_0-\bar{\delta}-\hat{\delta}).$$

By (6.22) we get

$$\begin{aligned}\mathcal{D} \;&= \Big(\quad (2-\alpha_0-\beta_0) + \Delta_x\Big(\mu(\beta_0-\alpha_0) - c_0 - c_1\Big) &(6.23)\\ &\quad +\Delta_x^2\left(\frac{\mu^2}{2}(2-\alpha_0-\beta_0) + \mu(c_1-c_0)\right) + o(\Delta_x^2)\Big)^2 \\ &- \quad 4\Big(1-\alpha_0-\beta_0-\Delta_x(c_0+c_1)\Big) &(6.24)\end{aligned}$$

It is clear that $\mathcal{D}$ admits the following asymptotic expansion, as $\Delta_x \to 0$:

$$\mathcal{D} = A_0 + A_1\Delta_x + A_2\Delta_x^2 + o(\Delta_x^2)$$

It is usefull to note that α_0 and β_0 can be expressed in terms of η_0 and ρ_0:

$$\alpha_0 = \frac{1-\eta_0-\rho_0}{2} \quad \text{and} \quad \beta_0 = \frac{1+\eta_0-\rho_0}{2}.$$

Let us compute A_0, A_1 and A_2 using standard analysis:

$$A_0 = (2-\alpha_0-\beta_0)^2 - 4(1-\alpha_0-\beta_0) = \alpha_0^2 + \beta_0^2 + 2\alpha_0\beta_0 = (\alpha_0+\beta_0)^2 = (1-\rho_0)^2.$$

$$\begin{aligned}A_1 &= 2(2-\alpha_0-\beta_0)\Big(\mu(\beta_0-\alpha_0) - (c_0+c_1)\Big) + 4(c_0+c_1)\\ &= 2\mu(2-\alpha_0-\beta_0)(\beta_0-\alpha_0) - 4(c_0+c_1) + 2(\alpha_0+\beta_0)(c_0+c_1) + 4(c_0+c_1)\\ &= 2\Big\{\mu(2-\alpha_0-\beta_0)(\beta_0-\alpha_0) + (\alpha_0+\beta_0)(c_0+c_1)\Big\}\\ &= 2\Big(\mu\eta_0(1+\rho_0) + c(1-\rho_0)\Big). &(6.25)\end{aligned}$$

$$\begin{aligned}A_2 &= 2(2-\alpha_0-\beta_0)\Big(\frac{\mu^2}{2}(2-\alpha_0-\beta_0) + \mu(c_1-c_0)\Big) + \Big(\mu(\beta_0-\alpha_0) - (c_0+c_1)\Big)^2\\ &= \mu^2\Big((2-\alpha_0-\beta_0)^2 + (\beta_0-\alpha_0)^2\Big)\\ &\quad +2\mu\Big((2-\alpha_0-\beta_0)(c_1-c_0) - (\beta_0-\alpha_0)(c_0+c_1)\Big) + (c_0+c_1)^2\\ &= 2\mu^2\Big((\alpha_0-1)^2 + (\beta_0-1)^2\Big) + 4\mu\Big((1-\beta_0)c_1 - (1-\alpha_0)c_0\Big) + (c_0+c_1)^2\\ &= \mu^2(\eta_0^2 + (1+\rho_0)^2) + 2\mu\Big((1+\rho_0)\overline{c} - \eta_0 c\Big) + c^2. &(6.26)\end{aligned}$$

Under the condition $\rho_0 \neq 1$, we have

$$\sqrt{\mathcal{D}} = (1-\rho_0)\sqrt{1 + \frac{A_1}{(1-\rho_0)^2}\Delta_x + \frac{A_2}{(1-\rho_0)^2}\Delta_x^2 + o(\Delta_x^2)}$$

Hence

$$\sqrt{\mathcal{D}} = B_0 + B_1\Delta_x + B_2\Delta_x^2 + o(\Delta_x^2)$$

with

$$B_0 = 1-\rho_0,$$

$$\begin{aligned}B_1 &= \frac{1}{2}\frac{A_1}{1-\rho_0} = \frac{1}{1-\rho_0}\Big\{\mu(2-\alpha_0-\beta_0)(\beta_0-\alpha_0) + (\alpha_0+\beta_0)(c_0+c_1)\Big\}\\ &= \mu\,\frac{\eta_0(1+\rho_0)}{1-\rho_0} + c\end{aligned}$$

$$B_2 \quad = \quad \frac{1}{2}\frac{A_2}{1-\rho_0} - \frac{1}{8}\frac{A_1^2}{(1-\rho_0)^3}. \tag{6.27}$$

As a result, B_2 is a second order polynomial function with respect to the μ-variable:

$$B_2 = \mu^2 B_{22} + \mu B_{21} + B_{20}.$$

Identities (6.25), (6.26) and (6.27) imply:

$$B_{20} = \frac{c^2}{2(1-\rho_0)} - \frac{\Big(2c(1-\rho_0)\Big)^2}{8(1-\rho_0)^3} = 0$$

$$B_{21} = \frac{1}{2}\frac{2\Big((1+\rho_0)\overline{c} - \eta_0 c\Big)}{1-\rho_0} - \frac{1}{8}\frac{8\eta_0 c(1-\rho_0)(1+\rho_0)}{(1-\rho_0)^3} = \overline{c}\frac{1+\rho_0}{1-\rho_0} - \frac{2\eta_0 c}{(1-\rho_0)^2}$$

$$\begin{aligned} B_{22} &= \frac{1}{2}\frac{\eta_0^2 + (1+\rho_0)^2}{1-\rho_0} - \frac{1}{8}\frac{4\eta_0^2(1+\rho_0)^2}{(1-\rho_0)^3} = \frac{1}{2}\frac{\Big(\eta_0^2 + (1+\rho_0)^2\Big)(1-\rho_0)^2 - \eta_0^2(1+\rho_0)^2}{(1-\rho_0)^3} \\ &= \frac{(1+\rho_0)^2}{2(1-\rho_0)} - \frac{2\eta_0^2\rho_0}{(1-\rho_0)^3}. \end{aligned}$$

Consequently

$$\begin{aligned} \sqrt{\mathcal{D}} \quad &= \quad 1-\rho_0 + \Big(\mu\,\frac{\eta_0(1+\rho_0)}{1-\rho_0} + c\Big)\Delta_x + \Big\{\mu^2\Big(\frac{(1+\rho_0)^2}{2(1-\rho_0)} - \frac{2\eta_0^2\rho_0}{(1-\rho_0)^3}\Big) \\ &\quad + \mu\Big(\overline{c}\,\frac{1+\rho_0}{1-\rho_0} - \frac{2\eta_0 c}{(1-\rho_0)^2}\Big)\Big\}\Delta_x^2 + o(\Delta_x^2). \end{aligned} \tag{6.28}$$

2) The first order development suffices to determine the limit of $\phi(\mu,t)$ as $\Delta_x \to 0$. Indeed

$$\sqrt{\mathcal{D}} \quad = \quad 1-\rho_0 + \frac{\Delta_x}{1-\rho_0}\Big\{\mu\eta_0(1+\rho_0) + c(1-\rho_0)\Big\} + o(\Delta_x). \tag{6.29}$$

From (6.14) and (6.16) we can easely deduce

$$\vartheta_\pm \quad = \quad \frac{1}{2}(1+\rho_0) + \frac{\Delta_x}{2}(\mu\eta_0 - c) \pm \frac{1}{2}\Big\{1-\rho_0 + \Delta_x\Big(\frac{\mu\eta_0(1+\rho_0)}{1-\rho_0} + c\Big)\Big\} + o(\Delta_x).$$

Then

$$\vartheta_+ = 1 + \Delta_x\frac{\mu\eta_0}{1-\rho_0} + o(\Delta_x) \quad \text{and} \quad \vartheta_- = \rho_0 - \Delta_x\Big(\frac{\mu\eta_0\rho_0}{1-\rho_0} + c\Big) + o(\Delta_x). \tag{6.30}$$

Let $t' = \lfloor \frac{t}{\Delta_t} \rfloor \Delta_t$. Since $\tilde{Z}_t^\Delta = \tilde{Z}_{t'}^\Delta + (t-t')\Delta_x Y_{\lfloor t/\Delta_t \rfloor + 1}$ and $|Y_n| \le 1$, then

$$|\phi(\mu,t) - \phi(\mu,t')| \le C\Delta_x\Delta_t, \tag{6.31}$$

where C is a constant which only depends on μ.
Recall that identity (6.20) and Proposition 6.4 lead to

$$\phi(\mu,t') = a_+\vartheta_+^{t'/\Delta_t} + a_-\vartheta_-^{t'/\Delta_t} \tag{6.32}$$

where

$$a_+ = \frac{(1-\alpha)e^{2\mu\Delta_x} + \alpha - \vartheta_- e^{\mu\Delta_x}}{\sqrt{\mathcal{D}}} \qquad \text{and} \quad a_- = e^{\mu\Delta_x} - a_+. \tag{6.33}$$

It is obvious that (6.33) and (6.30) imply: $\lim_{\Delta_x\to 0} a_+ = 1$ and $\lim_{\Delta_x\to 0} a_- = 0$.
Since $\lim_{\Delta_t\to 0}\vartheta_- = \rho_0$ and $-1 < \rho_0 < 1$ then

$$\lim_{\Delta_x\,\Delta_t\to 0} a_-\vartheta_-^{t'/\Delta_t} = 0. \tag{6.34}$$

Consequently, the second term in (6.32) tends to 0. It is important to note that the initial condition $X_0 = Y_0 = -1$ disappears. Let us study the first term in the right hand side of

(6.32). Note that $\lim_{\Delta_x\to 0}\vartheta_+ = 1$, then if Δ_x is small enough, we can take the logarithm of ϑ_+. From (6.30) a straightforward calculation gives

$$\log\vartheta_+ = \Delta_x\frac{\mu\eta_0}{1-\rho_0} + o(\Delta_x)$$

Choosing $r\Delta_t = \Delta_x$ and using (6.32), (6.34) and (6.31), we obtain the following limit:

$$\lim_{\Delta_x\to 0}\phi(\mu,t) = \exp\{\frac{r\mu\eta_0 t}{1-\rho_0}\}.$$

Since the convergence holds for any $\mu\in\mathbb{R}$, we can conclude (cf Theorem 3 in [4]) that

$$\lim_{\Delta_x\to 0}\mathbb{E}_{-1}[\exp(iu\tilde{Z}_t^{\Delta})] = \exp\Big\{-\frac{iur\eta_0 t}{1-\rho_0}\Big\},\quad \text{for any } u\in\mathbb{R}.$$

Thus $\tilde{Z}_t^{\Delta}$ converges in distribution, as $\Delta_x\to 0$, to the Dirac measure at $-\frac{r\eta_0 t}{1-\rho_0}$.
3) Next, we consider the convergence of the process

$$\xi_t^{\Delta} = \tilde{Z}_t^{\Delta} + \frac{t\eta_0\sqrt{r}}{(1-\rho_0)\sqrt{\Delta_t}}.$$

Hence we define

$$\psi(\mu,t) = \mathbb{E}_{-1}[e^{-\mu\xi_t^{\Delta}}] = e^{-\frac{\mu t\eta_0\sqrt{r}}{(1-\rho_0)\sqrt{\Delta_t}}}\phi(\mu,t).$$

To determine the limit of $\psi(\mu,t)$ as $\Delta_t,\Delta_x\to 0$, from (6.32) and (6.34) we may deduce that it suffices to compute the second order development of the root ϑ_+. Using (6.14) and (6.28) we get:

$$\begin{aligned}
\vartheta_+ &= \frac{1}{2}(1+\rho_0) + \frac{\Delta_x}{2}(\mu\eta_0 - c) + \frac{\Delta_x^2}{2}\Big(\frac{\mu^2(1+\rho_0)}{2} + \mu\bar{c}\Big)\\
&+ \frac{1-\rho_0}{2} + \frac{\Delta_x}{2}\Big(\frac{\mu\eta_0(1+\rho_0)}{1-\rho_0} + c\Big)\\
&\quad + \frac{\Delta_x^2}{2}\Big(\mu^2\Big(\frac{(1+\rho_0)^2}{2(1-\rho_0)} - \frac{2\eta_0^2\rho_0}{(1-\rho_0)^3}\Big) + \mu\Big(\bar{c}\frac{1+\rho_0}{1-\rho_0} - \frac{2\eta_0 c}{(1-\rho_0)^2}\Big)\Big) + o(\Delta_x^2).
\end{aligned}$$

As a result

$$\vartheta_+ = 1+\Delta_x\frac{\mu\eta_0}{1-\rho_0} + \Delta_x^2\Big(\frac{\mu^2}{2}\Big(\frac{1+\rho_0}{1-\rho_0} - \frac{2\eta_0^2\rho_0}{(1-\rho_0)^3}\Big) + \mu\Big(\frac{\bar{c}}{1-\rho_0} - \frac{\eta_0 c}{(1-\rho_0)^2}\Big)\Big) + o(\Delta_x^2). \quad (6.35)$$

We take $r\Delta_t = \Delta_x^2$. Then

$$\begin{aligned}
\lim_{\Delta_x\to 0}\psi(\mu,t) &= \lim_{\Delta_x\to 0}\Big(a_+\vartheta_+^{rt/\Delta_x^2}\exp\Big\{-\frac{\mu r\eta_0 t}{1-\rho_0}\frac{1}{\Delta_x}\Big\}\Big)\\
&= \lim_{\Delta_x\to 0}\exp\Big\{-\frac{\mu r\eta_0 t}{1-\rho_0}\frac{1}{\Delta_x} + \frac{rt}{\Delta_x^2}\log\vartheta_+\Big\}.
\end{aligned}$$

It is straightforward to deduce

$$\lim_{\Delta_x\to 0}\psi(\mu,t) = \exp\Big\{-m\mu + \frac{\sigma^2\mu^2}{2}\Big\} \quad (6.36)$$

with

$$m = r\Big(\frac{-\bar{c}}{1-\rho_0} + \frac{\eta_0 c}{(1-\rho_0)^2}\Big)t \quad (6.37)$$

$$\sigma^2 = \frac{r(1+\rho_0)}{1-\rho_0}\Big(1 - \frac{\eta_0^2}{(1-\rho_0)^2}\Big)t. \quad (6.38)$$

4) Since (6.36) holds for any $\mu\in\mathbb{R}$, this implies that ξ_t^{Δ} converges in distribution, as $\Delta_x\to 0$, to the Gaussian distribution with mean m and variance σ^2. (see Theorem 3 in [4]) □

Proposition 6.6. *Assume that $\rho_0 \neq 1$ and $r\Delta_t = (\Delta_x)^2$. Let us denote ξ^Δ the process defined by*

$$\xi_t^\Delta = \tilde{Z}_t^\Delta + \frac{t\sqrt{r}\eta_0}{(1-\rho_0)\sqrt{\Delta_t}}.$$

Then $(\xi_{t_1}^\Delta, \xi_{t_2}^\Delta, \ldots, \xi_{t_n}^\Delta)$ converges in distribution, as $\Delta_x \to 0$, towards $(\xi_{t_1}^0, \xi_{t_2}^0, \ldots, \xi_{t_n}^0)$ where ξ^0 is given by

$$\xi_t^0 = r\Big(\frac{-\bar{c}}{1-\rho_0} + \frac{\eta_0 c}{(1-\rho_0)^2}\Big)t + \sqrt{\frac{r(1+\rho_0)}{1-\rho_0}\Big(1 - \frac{\eta_0^2}{(1-\rho_0)^2}\Big)}W_t.$$

$(W_t,\, t \geq 0)$ is the one-dimensional Brownian motion starting at 0.

Proof. The proof is only presented in the case $n = 2$. For simplicity let $s = t_1 < t_2 = t$. We are interested in the limit of the random vector $(\xi_s^\Delta, \xi_t^\Delta)$. Let us then compute the two dimensional Fourier transform

$$\Psi^\Delta(\mu, \lambda) = \mathbb{E}_{-1}\left[e^{i\mu(\xi_t^\Delta - \xi_s^\Delta)} e^{i\lambda\xi_s^\Delta}\right], \quad (\lambda, \mu \in \mathbb{R}).$$

Since the process (X_t, Y_t) is Markovian, we obtain

$$\begin{aligned}\Psi^\Delta(\mu, \lambda) &= \mathbb{E}_{-1}\left[e^{i\mu\xi_{t-s}^\Delta}\right]\mathbb{E}_{-1}\left[1_{\{Y(s/\Delta_t)=-1\}}e^{i\lambda\xi_s^\Delta}\right] \\ &+ \mathbb{E}_{+1}\left[e^{i\mu\xi_{t-s}^\Delta}\right]\mathbb{E}_{-1}\left[1_{\{Y(s/\Delta_t)=+1\}}e^{i\lambda\xi_s^\Delta}\right],\end{aligned}$$

when s/Δ_t and t/Δ_t belongs to $\mathbb{N}$.
Note that $|\xi_u^\Delta - \xi_{u'}^\Delta| \leq \Delta_x\Delta_t$ when $u' = \left\lfloor \frac{u}{\Delta_t} \right\rfloor \Delta_t$. Consequently

$$\begin{aligned}&\Psi^\Delta(\mu, \lambda) \underset{\Delta_x \to 0}{\sim} \mathbb{E}_{-1}\left[e^{i\mu\xi_{t'-s'}^\Delta}\right]\mathbb{E}_{-1}\left[1_{\{Y(s'/\Delta_t)=-1\}}e^{i\lambda\xi_{s'}^\Delta}\right] \\ &+\mathbb{E}_{+1}\left[e^{i\mu\xi_{t'-s'}^\Delta}\right]\mathbb{E}_{-1}\left[1_{\{Y(s'/\Delta_t)=+1\}}e^{i\lambda\xi_{s'}^\Delta}\right], \quad (s' = \lfloor s/\Delta_t \rfloor\Delta_t,\ t' = \lfloor t/\Delta_t \rfloor \Delta_t).\end{aligned}$$

According to Proposition 2.2,

$$\lim_{\Delta_x \to 0} E_{-1}\left[e^{i\mu\xi_{t'-s'}^\Delta}\right] = \lim_{\Delta_x \to 0} E_{+1}\left[e^{i\mu\xi_{t'-s'}^\Delta}\right] = e^{(i\mu m - \frac{\sigma^2}{2}\mu^2)(t-s)}$$

where m and σ^2 are defined by (6.37), resp. (6.38). Then we can deduce:

$$\begin{aligned}\lim_{\Delta_x \to 0}\Psi^\Delta(\mu, \lambda) &= e^{(i\mu m - \frac{\sigma^2}{2}\mu^2)(t-s)}\lim_{\Delta_x \to 0}\mathbb{E}_{-1}\left[e^{i\lambda\xi_{s'}^\Delta}\right] \\ &= e^{(i\mu m - \frac{\sigma^2}{2}\mu^2)(t-s)}\lim_{\Delta_x \to 0}\mathbb{E}_{-1}\left[e^{i\lambda\xi_s^\Delta}\right] \\ &= e^{(i\mu m - \frac{\sigma^2}{2}\mu^2)(t-s)}e^{(i\lambda m - \frac{\sigma^2}{2}\lambda^2)s} \\ &= \mathbb{E}\left[\exp\{i\mu(\xi_t^0 - \xi_s^0) + i\lambda\xi_s^0\}\right]\end{aligned}$$

□

We are now able to end the proof of Theorem 2.2 (item 2). We may apply, without any change, the measure tension criterium used in the proof of convergence of (Z_t^Δ) in the case $\alpha_0 = \beta_0 = 1$ (see the end of Section 4). This, and Proposition 6.6 show that $(\xi_t^\Delta)_{t\geq 0}$ converges in distribution as $\Delta_x \to 0$ to the Brownian motion with drift $(\xi_t^0)_{t\geq 0}$.

6.3 Proof of Proposition 2.3

We suppose $\alpha_0 = \beta_0 = 1$, $c_1 = c_0 < 0$ and $r\Delta_t = \Delta_x^3$ where $r > 0$.
We briefly sketch the proof of Proposition 2.3. The approach is similar to the one developed in the case 2) of Theorem 2.2. We only prove that $\tilde{Z}_t^\Delta$ converges to the Gaussian distribution with 0-mean and variance equals $-rc_0t$. Using Theorem 3 in [4], it is equivalent to show

$$\lim_{\Delta_x \to 0}\mathbb{E}_{-1}\left[e^{-\mu\tilde{Z}_t^\Delta}\right] = e^{\frac{-rc_0t\mu^2}{2}}, \quad \forall \mu \in \mathbb{R}.$$

We have already observed that we may reduce to the case $t/\Delta_t \in \mathbb{N}$; in this case we have $\tilde{Z}_t^\Delta = Z_t^\Delta$ and

$$\mathbb{E}_{-1}\left[e^{-\mu Z_t^\Delta}\right] = \Phi\Big(e^{-\mu\Delta_x}, \frac{t}{\Delta_t}\Big)$$

where $\Phi(\lambda, t)$ is the moment generating function associated with (X_t) (see the beginning of subsection 6.1). Recall that $\Phi(\lambda, t)$ is given by identity (6.15).
Note that:

$$\alpha = \alpha_0 + c_0\Delta_x = 1 + c_0\Delta_x, \quad \beta = \beta_0 + c_0\Delta_x = 1 + c_0\Delta_x.$$

Since α and β have to belong to $[0, 1]$, this implies that $c_0 < 0$. Recall that $\mathcal{D}$, ϑ_+ and ϑ_- are the real numbers which have been defined by (6.12) resp. (6.14) (with $\lambda = e^{-\mu\Delta_x}$). We have:

$$\mathcal{D} = 4c_0^2\Delta_x^2\cosh^2(\mu\Delta_x) + 4(1 + 2c_0\Delta_x),$$

$$\vartheta_\pm = -c_0\Delta_x\cosh(\mu\Delta_x) \pm \sqrt{c_0^2\Delta_x^2\cosh^2(\mu\Delta_x) + 1 + 2c_0\Delta_x}.$$

Using classical analysis we get:

$$\sqrt{\mathcal{D}}/2 \quad = \quad \sqrt{1 + 2c_0\Delta_x + c_0^2\Delta_x^2 + o(\Delta_x^3)} = 1 + c_0\Delta_x + o(\Delta_x^3),$$

$$\vartheta_+ = 1 - \frac{c_0\mu^2}{2}\Delta_x^3 + o(\Delta_x^3), \quad \vartheta_- = -1 - 2c_0\Delta_x + o(\Delta_x).$$

$$\lim_{\Delta_x\to 0} a_+ = 1, \quad \lim_{\Delta_x\to 0}\vartheta_+^{t/\Delta_t} = \lim_{\Delta_x\to 0}\exp\Big\{-\frac{t}{\Delta_t}\frac{c_0\mu^2}{2}\Delta_x^3\Big\} = \exp\Big\{-c_0 r\frac{\mu^2}{2}t\Big\},$$

$$\lim_{\Delta_x\to 0} a_- = 0, \quad \lim_{\Delta_x\to 0}|\vartheta_-|^{t/\Delta_t} = \lim_{\Delta_x\to 0}\exp\Big\{\frac{t}{\Delta_t}2c_0\Delta_x\Big\} = \lim_{\Delta_x\to 0}\exp\Big\{\frac{2c_0 rt}{\Delta_x^2}\Big\} = 0 \quad (c_0 < 0).$$

Relation (6.15) implies that the variable Z_t^Δ is asymptotically normal distributed with variance $-rc_0t$.

References

[1] Xavier Bardina and Maria Jolis. Weak approximation of the Brownian sheet from a Poisson process in the plane. *Bernoulli*, 6(4):653–665, 2000.

[2] Patrick Billingsley. *Convergence of probability measures.* Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[3] M. Brancovan and T. Jeulin. *Probabilités - Cours et exercices corrigés.* Mathématiques l'université. Ellipses, première edition, 2006.

[4] J. H. Curtiss. A note on the theory of moment generating functions. *Ann. Math. Statistics*, 13:430–433, 1942.

[5] Eugene C. Eckstein, Jerome A. Goldstein, and Mark Leggas. The mathematics of suspensions: Kac walks and asymptotic analyticity. In *Proceedings of the Fourth Mississippi State Conference on Difference Equations and Computational Simulations (1999)*, volume 3 of *Electron. J. Differ. Equ. Conf.*, pages 39–50 (electronic), San Marcos, TX, 2000. Southwest Texas State Univ.

[6] Nathanaël Enriquez. Correlated processes and the composition of generators. In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 329–342. Springer, Berlin, 2007.

[7] R. Fürth. *Schwankungerscheinungen in der Physik.* Sammlung Vieweg, Braunschweig, 1920.

[8] S. Goldstein. On diffusion by discontinuous movements, and on the telegraph equation. *Quart. J. Mech. Appl. Math.*, 4:129–156, 1951.

[9] Richard Griego and Reuben Hersh. Theory of random evolutions with applications to partial differential equations. *Trans. Amer. Math. Soc.*, 156:405–418, 1971.

[10] Urs Gruber and Martin Schweizer. A diffusion limit for generalized correlated random walks. *J. Appl. Probab.*, 43(1):60–73, 2006.

[11] Eric Renshaw and Robin Henderson. The correlated random walk. *J. Appl. Probab.*, 18(2):403–414, 1981.

[12] Sheldon M. Ross. *Introduction to probability models. 8th ed.* Amsterdam: Academic Press. xvii, 2003.

[13] Daniel W. Stroock. *Lectures on topics in stochastic differential equations*, volume 68 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Tata Institute of Fundamental Research, Bombay, 1982. With notes by Satyajit Karmakar.

[14] G. I. Taylor. Diffusion by continuous movements. *Proceedings of the London Mathematical Society*, 20:196–212, 1921/22.

[15] P. Vallois and C. S. Tapiero. Memory-based persistence in a counting random walk process. *Physica A*, 386:303–317, 2007.

[16] P. Vallois and C. S. Tapiero. A claims persistence process and insurance. *Submitted to Insurance: Economics and Mathematics*, 2008.

[17] George H. Weiss. *Aspects and applications of the random walk*. Random Materials and Processes. North-Holland Publishing Co., Amsterdam, 1994.

[18] George H. Weiss. Some applications of persistent random walks and the telegrapher's equation. *Phys. A*, 311(3-4):381–410, 2002.