Boundedness and convergence of some self-attracting diffusions

Samuel Herrmann and Bernard Roynette

Institut de Mathématiques Elie Cartan, Université Henri Poincaré,
B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France
e-mail: {Samuel.Herrmann, Bernard.Roynette}@iecn.u-nancy.fr

Abstract: We consider some self-attracting diffusions and study the behaviour of their paths when time tends to infinity. We prove that, in case the interaction is nonlocal, the paths are bounded a.s. even if they don't converge. Otherwise, we generalize the convergence result of M. Cranston and Y. Le Jan.

2000 Mathematics Subject Classifications: 60G17, 60J60, 60K35

Keywords: comparison result, Girsanov theorem, long memory process

Introduction

Let B be an one-dimensional Brownian motion and $\Phi : \mathbb{R} \to \mathbb{R}$ be odd and bounded. In this paper we will be concerned with self-attracting diffusions, that is processes of the form

$$\begin{cases} dZ_t = dB_t - \left(\int_0^t \Phi(Z_t - Z_s) ds \right) dt \\ Z_0 = 0. \end{cases} \tag{1}$$

For Φ measurable, (1) has a unique solution in the sense of probability laws. If we assume that Φ is Lipschitz-continuous, (1) has a unique strong solution. The aim of this paper is to study the behaviour of this solution when $t \to \infty$. In [C-LJ] two cases were studied: the linear interaction where Φ is a linear function and the "constant" interaction in dimension 1. In both cases the authors proved that the diffusions converge a.s. O. Raimond studied the "constant" self-attraction for dimension larger than two: $\Phi(x) = \frac{ax}{\|x\|}$ with $a > 0$, and emphasises the same result ([Ra]). In case the interaction is constant, the convergence is proved thanks to a suitable comparison result and using the fact that Z_t jointly with its occupation measure $\mu_t = \int_0^t \delta_{Z_s} ds$ is a Markov process.

The first aim of this paper is to present one dimensional convergence and boundedness result for other functions Φ. After exploring questions of existence and uniqueness for solutions to the S.D.E. (1) in Sect. 1, we study, in the second section, the case of an odd, increasing and bounded function Φ verifying in a neighbourhood of the origin

$$|\Phi(x)| \geq C \exp -\frac{\rho}{|x|^k},$$

where C and ρ are positive constants and $k \in \mathbb{N}^*$. Under this assumption, we shall prove that Z_t converges a.s.

Our goal in the last section, which contains the most important result of this paper, is to explore the nonlocal interaction case. Even if the preceding assumption is really weak, the results are completely different as these obtained in the nonlocal case. In fact we shall consider the interaction function $\Phi(x) = \text{sgn}(x)\mathbb{1}_{\{|x|\geq a\}}$, with $a > 0$. M. Cranston and Y. Le Jan have already underline that the paths of this diffusion do not converge a.s. (see [C-LJ]). Nevertheless we shall prove, in order to complete their study, that the paths of this self-attracting diffusion are a.s. bounded.

1 Existence and uniqueness

In this section we explore briefly questions of existence and uniqueness for solutions to the stochastic differential equation (1).

Proposition 1 (existence and uniqueness)

- *If Φ is a measurable bounded function then there exists a unique weak solution to the S.D.E.* (1)*,*
- *moreover, if Φ is continuous then there exists at most one strong solution. If Φ is a Lipschitz function, then it exists a unique strong solution.*

Proof : The demonstration is based on a generalisation of the Girsanov theorem (see Corollary 3.5.2 in [K-S]) which implies the existence of a unique weak solution to the equation (1). For the existence and uniqueness of a strong solution, we use essentially Corollary 1 p.271 in [G-S] and Theorem 1.1 p.163 in [I-W]. The details of this proof are left to the reader. **QED**

Let us note that $Z_t^x = x + Z_t$ is the unique weak solution of equation (1) with initial condition $Z_0^x = x$. It is then reasonable to study only the solutions starting from the origin.

2 The support of Φ contains a neighbourhood of the origin

Our intent in this section is to study the behaviour of the sample paths of the solution of (1) in case Φ is an odd increasing bounded and continuous function which support contains a neighbourhood of the origin. In that case, under suitable conditions, the paths of the unique strong solution converge a.s. The growth of the function Φ is of prime importance: if we consider an odd function, decreasing on $\mathbb{R}_+^*$ and such that $\lim_{x\to\infty} \Phi(x) = 0$, we are not able to prove the convergence of the sample paths using the same way; we obtain only the boundedness of the paths (see [H]).

Theorem 1 *If there exist $C > 0$, $\rho > 0$ and $k \in \mathbb{N}^*$ such that*

$$|\Phi(x)| \geq C \exp -\frac{\rho}{|x|^k}, \tag{2}$$

in a neighbourhood of the origin, then the paths of Z_t, solution to the S.D.E. (1), converge a.s. as $t \to \infty$.

Let us note that Theorem 1 is practically optimal, since the assumption (2) is really weak.
The procedure to prove Theorem 1 is exactly the same as that presented in the demonstration of Theorem 2 p.92 in [C-LJ]. All the arguments can be justified by Proposition 2 formulated with regard to the following S.D.E.:

$$\begin{cases} dY_t = dW_t - \left(\int_0^t \Phi(Y_t - Y_s) ds + V(Y_t) \right) dt \\ Y_0 = 0 \end{cases} \tag{3}$$

where W is a Brownian motion and V is a measurable locally bounded fonction, nonnegative on $\mathbb{R}_+$. We denote by $\mathbb{P}^{(V)}$ the law of Y.

Proposition 2 *Under the assumptions of Theorem 1, we get*

- *a)* $\sup_{t \in \mathbb{R}_+} Y_t < \infty \quad \mathbb{P}^{(V)}$ *a.s.*
- *b) if $V(x) \geq \eta > 0$ when $x \geq 0$ then, for each $\varepsilon > 0$,*

$$\mathbb{P}^{(V)} \left(\sup_{t \in \mathbb{R}_+} Y_t > \varepsilon \right) \leq \varphi_\varepsilon(\eta)$$

where $\lim_{\eta \to \infty} \varphi_\varepsilon(\eta) = 0$.

Set $\mu_t = \int_0^t \delta_{Y_s} ds$. Clearly, (Y_t, μ_t) is a Markov process. Moreover, we have

Lemma 1 *For any stopping time τ, the conditional law with respect to $\mathbb{P}^{(V)}$ of $Y_{t+\tau} - Y_\tau$ given (Y_τ, μ_τ) coincides with $\mathbb{P}^{(V_\tau)}$, where*

$$V_\tau(x) = V(x + Y_\tau) + \int \Phi(x + Y_\tau - y)\mu_\tau(dy). \tag{4}$$

The proof of this lemma is straightforward.
Proof of Proposition 2:
i) Let us introduce a spatial scale $L_0 = 0$, $L_n = L_{n-1} + l_n$ where $l_n = R/n^2$ with $R > 0$. Let us denote $L_\infty = \lim_{n\to\infty} L_n$. Then we remark that $\lim_{R\to\infty} L_\infty = +\infty$.
We also define an other nonnegative sequence $(\alpha_n)_{n\geq 1}$ which tends to infinity. We shall determine this sequence more precisely in the following. Furthermore let us define the stopping time $S_a = \inf\{t > 0,\, Y_t = a\}$, for $a \in \mathbb{R}_+$, and the sequence $\tau_0 = 0$, $\tau_n = S_{L_n}$.
In order to prove that the paths of the solution Y are a.s. bounded above, it is enough to consider the following event

$$A := \bigcap_{n=0}^{\infty} \{\tau_{n+1} - \tau_n > \alpha_{n+1} \ \text{ or } \tau_n = \infty\},$$

and to prove that $\lim_{R\to\infty} \mathbb{P}(A^c) = 0$. Actually we get:

$$\{\sup Y_t < L_\infty\} \supseteq A.$$

By Lemma 1, we obtain

$$\begin{aligned} \mathbb{P}(A^c) &\leq \mathbb{P}(\tau_1 \leq \alpha_1) \\ &+ \sum_1^\infty \mathbb{E}\left[\prod_0^{n-1} \mathbb{1}_{\{\infty > \tau_{j+1} - \tau_j > \alpha_{j+1}\}} \mathbb{P}^{(V_n)}(S_{l_{n+1}} < \alpha_{n+1})\right] \end{aligned} \tag{5}$$

where $V_n = V_{\tau_n}$. Since Φ is increasing and since $V(x) \geq \eta$ for $x \geq 0$, using Lemma 1, we get:
Given $\cap_0^{n-1}\{\infty > \tau_{j+1} - \tau_j > \alpha_{j+1}\}$, for $t \in [\tau_n, \tau_{n+1} \wedge (\tau_n + \alpha_{n+1})]$ and $x \geq 0$, under $\mathbb{P}^{(V)}$,

$$V_n(x) \geq \sum_{j=0}^{n-1} \alpha_j \Phi\left(\sum_{k=j+1}^{n} l_k\right) + \eta$$

(we shall take $\eta = 0$ to prove a)). Then we define

$$\alpha_n = \frac{n^{k+3}}{\Phi\left(\frac{R}{2(n+1)}\right)}, \quad \text{for } n \geq 2,$$

and $\alpha_1 = 1/\Phi(R/4)$.
We denote by D_n a lower bound of the opposite of the drift on $[\tau_n, \tau_{n+1} \wedge (\tau_n + \alpha_{n+1})]$ and for $x \geq 0$. Then, since Φ is a increasing function, we get for $n \geq 2$,

$$\begin{aligned} D_n &\geq \sum_{j=1}^{\lfloor (n-1)/2 \rfloor} \alpha_j \Phi\left(\frac{R}{2(j+1)}\right) - \alpha_{n+1}\Phi\left(\frac{R}{(n+1)^2}\right) + \eta \\ &\geq \frac{1}{k+3}\left\lfloor \frac{n-1}{2} \right\rfloor^{k+4} - (n+1)^{k+3} + \eta := C_n + \eta. \end{aligned}$$

Let us also define $C_1 = -1$.
Then it appears that there exists $n_0(k) \in \mathbb{N}$ such that, for $n \geq n_0$, $C_n > 0$.
ii) Hence, using a comparison result (see, for instance [Ra] Lemma 2 p.179) and $D_n \geq C_n + \eta$, we obtain the following inequality

$$\mathbb{P}^{(V_n)}\left(S_{l_{n+1}} < \alpha_{n+1}\right) \leq \mathbb{Q}\left(S_{l_{n+1}} < \alpha_{n+1}\right), \tag{6}$$

where $\mathbb{Q}$ is the law of the continuous nonnegative process Z solution of the equation

$$\begin{cases} dZ_t = dB_t - (C_n + \eta)dt + dL_t \\ X_0 = 0, \end{cases}$$

L_t is an increasing process with finite variation, L_t is increasing only when $Z_t = 0$. Furthermore, for $\beta > 0$, we get

$$\mathbb{Q}\left(S_{l_{n+1}} < \alpha_{n+1}\right) \leq e^{\beta\alpha_{n+1}}\mathbb{E}_{\mathbb{Q}}[e^{-\beta S_{l_{n+1}}}]. \tag{7}$$

We are able to compute the right side of (7) using arguments like these formulated in Step 2 p.183 in [Ra]. By (6) and since Φ verifies the assumption of Theorem 1, we obtain, for $n \geq n_0$,

$$\mathbb{P}^{(V_n)}\left(S_{l_{n+1}} < \alpha_{n+1}\right) \leq \frac{2e}{C}(C_n + \eta + 1)^2 (n+1)^{k+3} e^{\frac{(2(n+2)/R)^k}{\rho} - 2(C_n+\eta)l_{n+1}}. \tag{8}$$

Let us remark that

$$\sum_{n \geq 0} \mathbb{P}^{(V_n)}\left(S_{l_{n+1}} < \alpha_{n+1}\right) < \infty,$$

since $(2(n+2)/R)^k$ is a polynomial on n of degree k, $(C_n+\eta)l_{n+1}$ is a polynomial of degree $k+2$ and $(C_n+\eta+1)^2(n+1)^{k+3}$ is a polynomial of degree $2k+7$.
iii) We shall end this proof by considering two different cases: $\eta = 0$ or $\eta > 0$. In order to prove a), let us choose $\eta = 0$ and $R > 1$. Then $((2n+4)/R)^k \leq (2n+4)^k$. Hence, the right side of (8) is less that an expression in which the only term depending on R is l_{n+1}. Hence, using the monotone convergence theorem we deduce that

$$\lim_{R\to\infty} \sum_{n\geq n_0} \mathbb{P}^{(V_n)}\left(S_{l_{n+1}} < \alpha_{n+1}\right) = 0.$$

Otherwise, for $1 \leq n \leq n_0 - 1$, $\mathbb{Q}\left(S_{l_{n+1}} < \alpha_{n+1}\right) \to 0$ when $R \to \infty$. By (5), we deduce a).
In the second case: $\eta > 0$, let us choose R small enough such that $L_\infty = \varepsilon$, and let tend $\eta \to \infty$. Then by the same arguments as these used in the case $\eta = 0$ we deduce b), using again the monotone convergence theorem. **QED**

3 Nonlocal interaction

In this section we are interesting in the study of the following S.D.E.:

$$X_t = B_t - \nu \int_0^t \int_0^s \operatorname{sgn}(X_s - X_u) \mathbb{1}_{\{|X_s - X_u| \geq \alpha\}} du ds, \quad \nu > 0, \tag{9}$$

where B is a one-dimensional Brownian motion. Let us recall that the sample paths of the diffusion converge a.s. provided $\alpha = 0$. Let us now assume $\alpha > 0$. Then the behaviour of the paths is completely different. In particular, we get the following

Lemma 2 (Cranston-Le Jan) *Let X be the solution of the S.D.E.* (9)*, then a.s. the paths do not converge.*

The aim of this section is to prove that, nevertheless, the paths are a.s. bounded.

3.1 Preliminary results:

First we provide a result with regard to the law of large numbers, afterwards we shall study the time spent by a diffusion in an interval.

Lemma 3 *Let $(A_i)_{i\geq 1}$ be a sequence of independent square integrable random variables such that*

- $\mathbb{E}[A_i] > 0$, *for all* $i \in \mathbb{N}^*$, *and* $\sum_{i=1}^{\infty} \mathbb{E}[A_i] = \infty$,
- $\sum_{i=1}^{\infty} Var(A_i) < \infty$,

then

$$\frac{\sum_{i=1}^{n} A_i}{\sum_{i=1}^{n} \mathbb{E}[A_i]} \to 1, \quad a.s.$$

Proof : Let us consider the random variable

$$B_n = \frac{A_n - \mathbb{E}[A_n]}{\sum_{i=1}^{n} \mathbb{E}[A_i]}.$$

Thus $\mathbb{E}[B_n] = 0$, for $n \geq 1$, and

$$\mathbb{E}[B_n^2] = \frac{Var(A_n)}{(\sum_{i=1}^{n} \mathbb{E}[A_i])^2}.$$

Since $\sum_{n\geq 1} \mathbb{E}[B_n] = 0$ and $\sum_{n\geq 1} \mathbb{E}[B_n^2] < \infty$, using Kolmogorov's lemma (see, for instance, Lemma 2.2.1 p.42 in [Re]), we get the a.s. convergence of $\sum_{n\geq 1} B_n$. Hence, it suffices to apply Kronecker's theorem (see Theorem 1.2.2 p.35 in [Re]) to get the a.s. convergence. QED

Let us study now the time spent by a diffusion in a suitable interval. We consider the following diffusion

$$X_t = B_t - \int_0^t n \mathbb{1}_{[0,1]}(X_s) ds, \tag{10}$$

stopped in τ which is the exit time of the interval $[-1, a]$, $0 < a \leq 1$.

Proposition 3 *Set*

$$\Gamma_1(\lambda) := \mathbb{E}\left[\exp\left\{-\lambda \int_0^{\tau} \mathbb{1}_{[0,a]}(X_s) ds\right\} \mathbb{1}_{X_\tau = -1}\right],$$

then

$$\Gamma_1(\lambda) = \frac{e^{2a\sqrt{n^2+2\lambda}} - 1}{n + \sqrt{n^2 + 2\lambda} - 1 - (n - \sqrt{n^2 + 2\lambda} - 1)e^{2a\sqrt{n^2+2\lambda}}}. \tag{11}$$

Proof : Let us consider the diffusion starting from $x \in [-1, a]$. Let us denote

$$u(x) = \mathbb{E}_x \left[\exp\left\{ -\lambda \int_0^\tau \mathbb{1}_{[0,a]}(X_s) ds \right\} \mathbb{1}_{X_\tau = -1} \right].$$

By Feynman-Kac's formula, u is a solution of the following system on the interval $[-1, a]$

$$\begin{cases} \frac{1}{2} u'' - n \mathbb{1}_{[0,a]} u' - \lambda \mathbb{1}_{[0,a]} u = 0, \\ u(-1) = 1 \quad \text{and} \quad u(a) = 0. \end{cases} \tag{12}$$

Hence

$$u(x) = Ax + A + 1, \quad \text{with} \quad A \in \mathbb{R}, \quad \text{on} \quad [-1, 0],$$

$$u(x) = \alpha e^{(n+\sqrt{n^2+2\lambda})x} + \beta e^{(n-\sqrt{n^2+2\lambda})x}, \quad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}, \quad \text{on} \quad [0, a].$$

Since $u(a) = 0$, $u(0_-) = u(0_+)$ and $u'(0_-) = u'(0_+)$, we can compute eeasely A, α and β. The determination of $u(0)$ leads to the statement of Proposition 3.
QED

Let us now assume that a depends on n (a_n). We shall study the behaviour of the time spent by the diffusion in the interval $[0, a_n]$, when n tends to infinity.

Corollary 1 *If $a_n \sim \frac{1}{n^\gamma}$ with $\gamma \in [0, 1[$ when $n \to \infty$, then*

$$\Gamma_2 := \mathbb{E}\left[\int_0^\tau \mathbb{1}_{[0,a_n]}(X_s) ds \mathbb{1}_{X_\tau=-1} \right] \sim \frac{1}{n}. \tag{13}$$

The proof of this corollary is straightforward: it is enough to compute $\frac{\partial}{\partial \lambda} \Gamma_1(\lambda)|_{\lambda=0}$. Let us note that the equivalent of Γ_2 does not depend on γ. It seems to be surprising but, in fact, it is reasonable since the diffusion spends its time essentially in the negativ positions and in a neighbourhood of the origin.

Proposition 4 *Let us consider the diffusion*

$$X_t = B_t - n \int_0^t \mathbb{1}_{\{X_s \geq 0\}} ds,$$

and set $\tau = \inf\{t > 0, \ X_t = -1\}$ then

$$\mathbb{E}\left[\exp -\lambda \int_0^\tau \mathbb{1}_{[0,+\infty)}(X_s) ds \right] = \frac{1}{1 + \sqrt{n^2 + 2\lambda} - n}. \tag{14}$$

The proof of this proposition is the same as the proof of Proposition 3. It suffices to define the system (12) on $[-1,+\infty)$ and to change the boundary values: $u(-1)=1$ and $u(\infty)=0$.

Corollary 2

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\tau \mathbb{1}_{[0,+\infty)}(X_s)ds\right] &= \frac{1}{n} \\
\mathbb{E}\left[\left(\int_0^\tau \mathbb{1}_{[0,+\infty)}(X_s)ds\right)^2\right] &= \frac{2n+1}{n^3}
\end{aligned}
\tag{15}
$$

The proof is straightforward.

3.2 Boundedness of the paths

Let us consider the solution of the following S.D.E.

$$
X_t = B_t - \int_0^t \int_0^s \operatorname{sgn}(X_s - X_u)\mathbb{1}_{\{|X_s-X_u|\geq 2\}}duds, \quad t \geq 0. \tag{16}
$$

Let us denote

$$
\Phi(x) = \operatorname{sgn}(x)\mathbb{1}_{\{|x|\geq 2\}}. \tag{17}
$$

Theorem 2 *The paths of the unique weak solution of* (16) *are a.s. bounded.*

We shall prove that the paths are bounded above a.s. Then we obtain the statement of the theorem by symmetry.
Proof : (throughout this proof, C_i will be some constants).
Set $\tau_n = \inf\{t \geq 0,\ X_t = n\},\ n \geq 4$. Let us consider the event $\Omega_n = \{\tau_n < \infty\}$ and assume that $\mathbb{P}(\Omega_n) > 0$ (otherwise the above boundedness of the paths is obvious). We shall prove that, conditionally to Ω_n, the probability that the path do not reach the level $n+2$ is positive and independent of n. That is enough to get the a.s. boundedness of the paths.
First, let us define the sequence $u_0 = 0$, $u_k = C_1 \times \{k(\log(k+1))^2\}^{-1}$ such that

$$
\sum_{k\geq 1} u_k = 1,
$$

and the sequences of stopping times $(S_k^{\pm})_{k\geq 0}$ and $(T_k)_{k\geq 0}$ by:

$$
S_0^{\pm} = \tau_n = T_0 \tag{18}
$$

$$
S_k^{-} = \inf\left\{t > 0, X_{t+U_{k-1}} = n - 1 + \sum_{j=0}^{k-1} u_j\right\}, \quad k \geq 1, \tag{19}
$$

$$S_k^+ = \inf\left\{t > 0, X_{t+U_{k-1}} = n + 1 + \sum_{j=0}^{k-1} u_j\right\}, \quad k \geq 1, \tag{20}$$

$$T_k = \inf\left\{t > 0, X_{t+W_k} = n + \sum_{j=0}^{k} u_j\right\}, \quad k \geq 1, \tag{21}$$

where

$$U_k = \sum_{j=0}^{k} S_j^- + \sum_{j=0}^{k} T_j \quad \text{and} \quad W_k = \sum_{j=0}^{k} S_j^- + \sum_{j=0}^{k-1} T_j. \tag{22}$$

(by convention $\inf \emptyset = +\infty$).

These sequences of stopping times are defined step by step. Let us describe

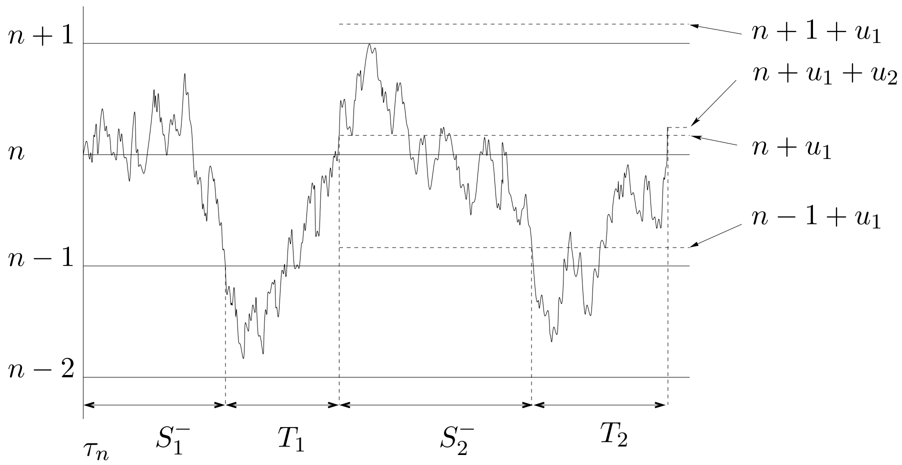

Figure 1: definition of different stopping times

step k. At time U_{k-1}, the diffusion X starts at the position $\left(n + \sum_{j=0}^{k-1} u_j\right)$, called "initial position". $S_k^{\pm}$ is then the time spent by the diffusion to reach the level (*initial position* ± 1). After time $U_{k-1} + S_k^-$ (i.e. W_k), the diffusion starting at (*initial position* $- 1$) uses a time T_k in order to reach the "initial position" shifted: $\left(n + \sum_{j=0}^{k-1} u_j + u_k\right)$. This level becomes then the initial level of step $k+1$ and

$$U_k = W_k + T_k = U_{k-1} + S_k^- + T_k.$$

In order to prove that the paths do not reach the level $n+2$ with a positive probability, it is enough to prove

$$\mathbb{P}\left(\bigcap_{k=1}^{\infty} E_k\right) > 0, \tag{23}$$

where

$$E_k = \{\omega \in \Omega_n \ \ s.t. \ S_j^- < S_j^+, \ 1 \le j \le k, \ \text{and} \ T_i < \infty, \ 1 \le i \le k-1\}. \quad (24)$$

Since the set of all paths verifying $T_k = \infty$ on $S_k^- < \infty$ for some $k \in \mathbb{N}^*$ are bounded above, it is enough to prove (23). Let us now prove this inequality in several steps.
step 1: Let us consider a stopping time T and define $Y_t = X_{t+T} - X_T$. Hence, by (16) and (17), we get

$$\begin{aligned}
Y_t &= B_{t+T} - B_T - \int_T^{T+t} ds \int_0^s \Phi(X_s - X_u) du \\
&= B_{t+T} - B_T - \int_0^t ds \int_0^s \Phi(X_{s+T} - X_{u+T}) du \\
&\quad - \int_0^t ds \int_0^T \Phi(X_{s+T} - X_u) du \\
&= \tilde{B}_t - \int_0^t ds \int_{\mathbb{R}} \Phi(Y_s + X_T - y) \mu_T(dy) + \int_0^t ds \int_0^s \Phi(Y_s - Y_u) du,
\end{aligned}$$

where $\tilde{B}$ is a Brownian motion independent of $\mathcal{F}_T$ ($\mathcal{F}_t = \sigma\{B_s, \ s \le t\}$) and $\mu_t = \int_0^t \delta_{X_s} ds$.
Let us denote $Y_t^{(k)} = X_{t+U_{k-1}} - X_{U_{k-1}}$. Then, on $[0, S_k^- \wedge S_k^+]$, we get

$$Y_t^{(k)} = \tilde{B}_t - \int_0^t \left(\int_{\mathbb{R}} \Phi \left(Y_s^{(k)} + n + \sum_{j=0}^{k-1} u_j - y \right) \mu_{U_{k-1}}(dy) \right) ds. \quad (25)$$

Actually $|Y_s^{(k)} - Y_u^{(k)}| \le 2$ for $0 \le u, s \le S_k^- \wedge S_k^+$. Let us note that, given $\mathcal{F}_{U_{k-1}}$, $Y^{(k)}$ is a diffusion which drift coefficient at time t only depends on $Y_t^{(k)}$. We also define $Z_t^{(k)} = X_{t+W_k} - X_{W_k}$ on $[0, T_k]$. Therefore we get

$$Z_t^{(k)} = \tilde{B}_t - \int_0^t \left(\int_{\mathbb{R}} \Phi \left(Z_s^{(k)} + n - 1 + \sum_{j=0}^{k-1} u_j - y \right) \mu_{W_k}(dy) \right) ds. \quad (26)$$

Let us now prove the inequality (23). Our goal is to find an upper bound of the probability $\mathbb{P}(S_k^- > S_k^+ | E_{k-1} \cap \{T_{k-1} < \infty\})$ (step 4). To obtain such a result, we shall compare the diffusion $Y^{(\cdot)}$ with an other simplier diffusion. This argument is based on a comparison result (see, for instance, Theorem 1.1 chapter 6 in [I-W]). That is why we shall find an upperbound of the drift of $Y^{(\cdot)}$. Since $Y^{(\cdot)}$ and $Z^{(\cdot)}$ are linked, it is enough to compare $Z^{(\cdot)}$ with

an other simplier diffusion, in other words to bound above the drift of $Z^{(\cdot)}$ (step 3). Since the drift of $Z^{(\cdot)}$ depends on the occupation time of $Y^{(\cdot)}$, the second step presents rough computations on $Y^{(\cdot)}$.
step 2: Let $0 < \gamma < 1$ and let us consider the following probability:

$$\mathcal{A}_m := \mathbb{P}\left(\int_0^{S_m^-} \mathbb{1}_{\{Y_s^{(m)} \geq 1 - \sum_{j=l}^{m-1} u_j\}} ds \geq m^\gamma \middle| E_m \right),$$

where $1 \leq l \leq m$. Using the Markov inequality we obtain, for $N \in \mathbb{N}^*$,

$$\mathcal{A}_m \leq \frac{1}{m^{\gamma N}} \mathbb{E}\left[\left(\int_0^{S_m^-} \mathbb{1}_{\{Y_s^{(m)} \geq 1 - \sum_{j=l}^{m-1} u_j\}} ds \right)^N \middle| E_m \right].$$

By (25) and since the drift of $Y^{(m)}$ is nonpositive for all $t \in [0, S_m^- \wedge S_m^+]$, we get $Y_t^{(m)} \geq \tilde{B}_t$ *a.s.*. We deduce that $\mathbb{P}(S_m^- < S_m^+) \geq 1/2$.
Hence

$$\mathcal{A}_m \leq \frac{2}{m^{\gamma N}} \mathbb{E}\left[\left(\int_0^{S_m^-} \mathbb{1}_{\{Y_s^{(m)} \geq 1 - \sum_{j=l}^{m-1} u_j\}} ds \right)^N \middle| E_{m-1} \cap \{T_{m-1} < \infty\} \right].$$

Let us compare $Y^{(m)}$ with the diffusion $\bar{Y}$ defined by the S.D.E.:

$$\bar{Y}_t = \tilde{B}_t - \tau_1 t.$$

We recall that τ_1 is the first time that X reaches the level 1 and that $\tilde{B}$ is independent of τ_1. Let us define $R_{-1} = \inf\{t > 0,\ \bar{Y}_t = -1\}$. Then we obtain

$$\mathcal{A}_m \leq \frac{2}{m^{\gamma N}} \mathbb{E}\left[\left(\int_0^{R_{-1}} \mathbb{1}_{\{\bar{Y}_s \geq 0\}} ds \right)^N \right] \leq \mathbb{E}[R_{-1}^N].$$

Since the law of R_{-1} conditionally to τ_1 and the law of τ_1 are known (see, for instance [B-S] formulas 2.0.2 p.163 and 2.0.2 p.223) we get

$$\mathcal{A}_m \leq \frac{1}{2\pi} \int_0^\infty \int_0^\infty x^{N-3/2} y^{-3/2} \exp\left\{ -\frac{1}{2y} - \frac{(yx-1)^2}{2x} \right\} dx\, dy.$$

Thus there exists a constant $K_N > 0$ such that, for all $N \geq 1$,

$$\mathcal{A}_m \leq \frac{K_N}{m^{\gamma N}}. \tag{27}$$

Let us now define the event F_k by:

$$F_k = \bigcap_{m=k-\lfloor k^\gamma \rfloor}^{k} G_m^k,$$

with

$$G_m^k = \left\{ \int_0^{S_m^-} \mathbb{1}_{\left\{Y_s^{(m)} \geq 1 - \sum_{j=k-\lfloor k^\gamma \rfloor -1}^{m-1} u_j\right\}} ds < m^\gamma \,\middle|\, E_{m-1} \cap \{T_{m-1} < \infty\} \right\}.$$

By (27),

$$\mathbb{P}(F_k^c) = \mathbb{P}\left(\bigcup_{m=k-\lfloor k^\gamma \rfloor}^{k} (G_m^k)^c \right) \leq \sum_{m=k-\lfloor k^\gamma \rfloor}^{k} \frac{K_N}{m^{\gamma N}}.$$

Since $\displaystyle\sum_{m=k-\lfloor k^\gamma \rfloor}^{k} \frac{K_N}{m^{\gamma N}} \sim \frac{C_2}{(k-\lfloor k^\gamma \rfloor)^{\gamma N - 1}}$ when k tends to infinity, we get, for N large enough,

$$\sum_{k \geq 1} \mathbb{P}(F_k^c) < \infty.$$

Therefore, using Borel-Cantelli's Lemma, we deduce that there exists a random variable $N_1(\omega) \in \mathbb{N}$ such that, for all $k \geq N_1(\omega)$, $\omega \in F_k$ *a.s.* Hence $\omega \in G_m^k$ *a.s.* for $k - \lfloor k^\gamma \rfloor \leq m \leq k$.

step 3: We are now able to find an upper bound of the drift of $Z^{(k)}$, for $k \geq N_1(\omega)$. By (26), on the interval $[0, T_k]$, the drift is equal to

$$b_k(x) := \int_{\mathbb{R}} \Phi\left(x + n - 1 + \sum_{j=1}^{k-1} u_j - y \right) \mu_{W_k}(dy) \quad \text{for} \quad x \in [-1, 1 + u_k].$$

Thus, given $E_k \cap \{T_k < \infty\}$, we get the following inequalities:

$$\begin{cases} b_k(x) \leq 0 \quad \text{if} \quad x \geq 0, \\ b_k(x) \leq \displaystyle\int_0^{S_k^-} \mathbb{1}_{\{Y_s^{(k)} \geq 1 - u_{k-1}\}} ds \quad \text{on} \quad [-u_{k-1}, 0[, \\ b_k(x) \leq \displaystyle\int_0^{S_{k-1}^-} \mathbb{1}_{\{Y_s^{k-1} \geq 1 - u_{k-2}\}} ds + \int_0^{S_k^-} \mathbb{1}_{\{Y_s^{(k)} \geq 1 - u_{k-1} - u_{k-2}\}} ds \\ \text{on} \quad [-u_{k-1} - u_{k-2}, -u_{k-1}[, \\ \vdots \end{cases}$$

Therefore, using the preceding results of step 2, we get

$$\begin{cases} b_k(x) \le k^\gamma \quad \text{on} \quad [-u_{k-1}, 0[, \\ b_k(x) \le k^\gamma + (k-1)^\gamma \quad \text{on} \quad [-u_{k-1} - u_{k-2}, -u_{k-1}[, \\ \vdots \\ b_k(x) \le k^\gamma + (k-1)^\gamma + \cdots + (k - \lfloor k^\gamma \rfloor)^\gamma \\ \text{on} \quad \left[-\sum_{j=k-\lfloor k^\gamma \rfloor -1}^{k-1} u_j, -\sum_{j=k-\lfloor k^\gamma \rfloor}^{k-1} u_j \right], \end{cases}$$

for $k \ge N_1(\omega)$. Set

$$\xi_k = \sum_{j=k-\lfloor k^\gamma \rfloor -1}^{k-1} u_j. \tag{28}$$

Let us note that, thanks to the definition of u_j ($u_j = C_1/(j(\log(j+1))^2)^{-1}$),

$$\begin{aligned} \xi_k &\ge C_1 \int_{k-\lfloor k^\gamma \rfloor -1}^{k-2} \frac{dx}{x(\log x)^2} = C_1 \int_{\log(k-\lfloor k^\gamma \rfloor -1)}^{\log(k-2)} \frac{du}{u^2} \\ &\ge C_1 \left(\frac{1}{\log(k - \lfloor k^\gamma \rfloor - 1)} - \frac{1}{\log(k-2)} \right) \ge C_3 \kappa_k, \end{aligned}$$

where

$$\kappa_k \sim \frac{1}{k^{1-\gamma} (\log k)^2} \quad \text{when} \quad k \to \infty. \tag{29}$$

Therefore, on $[-\xi_k, 0]$, the drift of $Z^{(k)}$ is bounded by

$$b_k(x) \le (k^\gamma + 1)^2. \tag{30}$$

Using the bound (30) of the drift of $Z^{(k)}$, we are able to estimate the time spent by the diffusion in the interval $(-\infty, 0]$. Given $E_k \cap \{T_k < \infty\}$ and thanks to a comparison result we get

$$\int_0^{T_k} \mathbb{1}_{[-1,0]}(Z_s^{(k)}) ds \ge \int_0^{R_1} \mathbb{1}_{[-1,0]}(\bar{Z}_s^{(k)}) ds =: A_k,$$

where

$$\bar{Z}_t^{(k)} = \tilde{B}_t + \int_0^t \left((k^\gamma + 1)^2 \mathbb{1}_{[-\xi_k, 0]}(\bar{Z}_s^{(k)}) + (W_k + s) \mathbb{1}_{(-\infty, -\xi_k[}(\bar{Z}_s^{(k)}) \right) ds.$$

Let us recall that ξ_k is defined by (28), W_k by (22). Moreover let us denote $R_a = \inf\{t > 0,\ \bar{Z}_t^{(k)} = a\}, \quad a \in \mathbb{R}$. Thus we deduce

$$\begin{aligned} \mathbb{E}[A_k] &\ge \mathbb{E}\left[\int_0^{R_1} \mathbb{1}_{[-1,0]}(\bar{Z}_s^{(k)}) ds \mathbb{1}_{\{\bar{Z}_s^{(k)} \ge -\xi_k,\ \forall s \in [0, R_1]\}} \right] \\ &\ge \mathbb{E}\left[\int_0^{R_1} \mathbb{1}_{[-1,0]}(\bar{Z}_s^{(k)}) ds \mathbb{1}_{\{R_1 < R_{-\xi_k}\}} \right]. \end{aligned}$$

Let us apply an adapted version of Corollary 1 (symmetry, n replaced by $k^{2\gamma}$). Since $\xi_k \geq \dfrac{C_4}{k^{1-\gamma_0}}$ forall $\gamma_0 < \gamma$, i.e. since $\xi_k \geq C_4 \left(\dfrac{1}{k^{2\gamma}}\right)^{1-\gamma_0-2\gamma}$ we obtain, for $\gamma > 1/3$,

$$\mathbb{E}[A_k] \geq \alpha_k \;\text{ where }\; \alpha_k \sim \frac{1}{k^{2\gamma}} \;\text{ in a neighbourhood of the infinity.} \tag{31}$$

Finally

$$\mathbb{E}[(A_k)^2] \leq \mathbb{E}\left[\left(\int_0^{R_1} \mathbb{1}_{[-1,0]}(\tilde{Z}_s^{(k)})ds\right)^2\right],$$

where $\tilde{Z}_t^{(k)}$ is the solution of

$$\tilde{Z}_s^{(k)} = \tilde{B}_t + \int_0^t (k^\gamma + 1)^2 \mathbb{1}_{(-\infty,0]}(\tilde{Z}_s^{(k)})ds.$$

By Corollary 2, we obtain the following inequality:

$$\mathbb{E}[(A_k)^2] \leq \frac{2(k^\gamma+1)^2+1}{(k^\gamma+1)^6}. \tag{32}$$

Hence

$$E[A_k] \leq \frac{C_5}{k^{2\gamma}}.$$

By (31), (32) and Lemma 3, we get, for $\gamma < 1/2$,

$$\sum_{j=1}^k A_j \sim \sum_{j=1}^k \frac{1}{i^{2\gamma}} \sim \frac{k^{1-2\gamma}}{1-2\gamma} \;\text{ a.s. when }\; k \to \infty.$$

Step 4: In this step we compute $\mathbb{P}(\bigcap_{k\geq 1} E_k)$, where E_k is given by (24). Let us choose $k \geq N_1(\omega)+1$ and let us bound above the drift $d_k(x)$ of $Y^{(k)}$. Let us recall that, in the computation of the drift of $Y^{(k)}$, there is no contribution of the time spent by the diffusion X in the interval $[x-1,x+1]$ since $\Phi(x) = \text{sgn}(x)\mathbb{1}_{\{|x|\geq 2\}}$. So, by (25), we get

$$d_k(x) \leq -\sum_{j=1}^m \int_0^{T_j} \mathbb{1}_{[-1,0]}(Z_s^{(j)})ds, \;\text{ for } 1 - \sum_{j=m-1}^{k-1} u_j \leq x \leq 1 - \sum_{j=m}^{k-1} u_j.$$

Thanks to the second step, for $m \geq N_1(\omega)$,

$$d_k(x) \leq -\sum_{j=N_1(\omega)}^m A_j =: -\beta_m \sim -\frac{m^{1-2\gamma}}{1-2\gamma} \; a.s. \tag{33}$$

when m and k tend to infinity. Therefore, using a comparison result, we obtain

$$\mathbb{P}(S_k^+ > S_k^- | E_{k-1} \cap \{T_{k-1} < \infty\} \cap \{k \geq N_1 + 1\}) \leq \frac{s_k(0) - s_k(-1)}{s_k(1) - s_k(-1)}, \quad (34)$$

s_k is the scale function of the diffusion $\bar{Y}^{(k)}$, solution of the S.D.E.

$$\bar{Y}_t^{(k)} = \tilde{B}_t - \int_0^t \bar{d}_k(\bar{Y}_s^{(k)}) ds,$$

with

$$\begin{cases} \bar{d}_k(x) = 0 \quad \text{on} \quad \left[-1, 1 - \sum_{j=N_1(\omega)}^{k-1} u_j\right], \\ \bar{d}_k(x) = -\beta_m \quad \text{on} \quad \left[1 - \sum_{j=m-1}^{k-1} u_j, 1 - \sum_{j=m}^{k-1} u_j\right], \quad m \geq N_1(\omega) + 1. \end{cases}$$

We compute then the scale function:

$$s_k(x) = \int_0^x \left(\exp -2 \int_0^\xi \bar{d}_k(y) dy\right) d\xi,$$

thus $s_k(x) = x$ for $x \leq 0$ and

$$s_k(1) \geq \frac{C_1}{k(\log(k+1))^2} \exp\left(2 \sum_{j=N_1}^{k-1} \frac{\beta_j}{j(\log(j+1))^2}\right).$$

Let us note that $\sum_{j=N_1}^{k-1} \frac{\beta_j}{j(\log(j+1))^2}$ tends to infinity when k becomes large. Moreover we get that, for $\theta > 0$ small enough, there exists a constant $C_6(\theta) > 0$ such that

$$s_k(1) \geq \frac{C_6(\theta)}{k(\log(k+1))^2} \exp\left(k^{1-2\gamma-\theta}\right).$$

By (34) and the equality $\frac{s_k(0) - s_k(-1)}{s_k(1) - s_k(-1)} = \frac{1}{s_k(1) + 1}$ we deduce that

$$\sum_{k \geq 1} \mathbb{P}(S_k^+ < S_k^- | E_{k-1} \cap \{T_{k-1} < \infty\}) < \infty,$$

and that the infinite sum is bounded by a constant which does not depend on n. Therefore, for $n \geq 4$, there exists a constant $\varepsilon \in]0, 1]$ independent of n such that $\mathbb{P}(\bigcap_{k \geq 1} E_k) \geq \varepsilon$. QED

References

[B-S] A.N. Borodin and P. Salminen, *Handbook of Brownian Motion -Facts and Formulae*, Birkhäuser Verlag, (1996).

[C-LJ] M. Cranston and Y. Le Jan, *Self-attracting diffusions : Two case studies*, Math. Ann.,**303** (1995), 87-93.

[G-S] I.Gihman and A.V. Skorohod, *Theory of stochastic processes III*, Springer Verlag, 1979.

[H] S. Herrmann, *Etude de processus de diffusion*, thesis, Institut Elie Cartan, Nancy.

[I-W] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd edition, North-Holland Kodansha, 1989.

[K-S] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, 2nd edition, Springer Verlag, 1991.

[P-V] R. Pemantle and S.Volkov, *Vertex-reinforced random walk on* $\mathbb{Z}$ *has finite range*, Ann. Probab., **27** (1999) no.3, pp. 1368-1388.

[Ra] O. Raimond, *Self Attracting Diffusions : Case of the constant interaction*, Probab. Theor. Relat. Fields **107**, (1996), 177-196.

[Re] P. Revesz, *The Laws of Large Numbers*, Probability and Mathematical Statistics, Academic Press New York and London, 1968.