Stochastic Resonance (365)

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Introduction

The concept of stochastic resonance was introduced by physicists. It originated in a toy model designed for a qualitative description of periodicity phenomena in the recurrences of glacial eras in Earth's history. It spread its popularity over numerous areas of natural sciences: neuronal response to periodic stimuli, variations of magnetization in a ferromagnetic system, voltage variations in the simple Schmitt trigger electronic circuit or in more complicated devices, behavior of lasers in optical bi-stability... The interest in this ubiquitous phenomenon is enhanced by signal analysis: an optimal dose of noise in some system can essentially boost signal transduction. Noise in this context does not enter the system as an impurity perturbing its performance, but on the contrary as a catalyst triggering amplified stochastic response to weak periodic signals.

1 The climate paradigm

The phenomenon of stochastic resonance was first discovered in an elementary climate model serving in an explanation of major transitions in paleoclimatic time series confining glacial cycles. Data collected for instance from ice or deep sea cores allow to deduce estimates of the average temperature on Earth over the last 700 000 years. They exhibit periodic switching between ice and warm ages with fast spontaneous transitions. The average periodicity of the glaciation time series obtained is about 10^5 years. In order to explain temperature variations, Benzi et al. [1] introduced random perturbations into an energy balance model of the Budyko-Sellers type. This model describes the evolution of the seasonal and global average temperature X caused by defects in the balance between incoming and outgoing radiation

$$c\,\frac{dX(t)}{dt} = E_{in} - E_{out},$$

where c is the active thermal inertia of the system. The incoming energy is modelled as proportional to the *solar constant* Q:

$$E_{in} = Q(1 + A\cos\frac{2\pi t}{T}), \quad \text{with } T \approx 92\,000 \text{ years}$$

and $A \approx 0.1\%$ of Q. This exceedingly small variation of the solar constant is caused by a modulation of the orbital eccentricity of the Earth's trajectory (Figure 1). The outgoing radiation E_{out} is composed of two essential parts. The first part $a(X)E_{in}$ is dominated by the albedo $a(X)$ representing the proportion of energy reflected back to space. It is a decreasing function of temperature, due to the higher rate of reflection from a brighter Earth at low temperatures implying a bigger volume of ice. The second part of the outgoing radiation comes from the fact that the Earth radiates energy like a black body, and is given by the Boltzmann law γX^4 where γ is the Stefan constant. Describing the balance of energy terms as a slowly and weakly time varying gradient of a potential U, the balance model can be expressed by

$$\frac{dX(t)}{dt} = -\frac{\partial U}{\partial x}\,(\frac{t}{T}, X(t))$$

where the time period 1 is blown up to (large) T by time scaling. The roles of deep and shallow wells switch periodically (Figure 2). Since the variation of the solar constant is extremely small, we can assume that the height of the barrier between the two wells is lower bounded by a positive constant. The system then admits three steady states two of which are stable and separated by roughly 10 K. As the solar constant, they fluctuate slowly and very weakly. Therefore this deterministic system cannot account for climate changes with temperature variations of about 10 K. They can only be explained by allowing transitions between the two steady states which become possible by adding noise to the system. In general short time scale phenomena such as annual fluctuations in solar radiation are modelled by Gaussian white noise of intensity ε and lead to equations of the type

$$dX_t^\varepsilon = -\frac{\partial U}{\partial x}(\frac{t}{T}, X_t^\varepsilon)dt + \sqrt{\varepsilon}dW_t, \tag{1.1}$$

which are generic for studying stochastic resonance in numerous physical and biological models. Generally, the input of noise amplifies a weak periodic signal by creating trajectories fluctuating randomly periodically between meta-stable states. An optimal tuning of noise intensity to period length (*stochastic resonance*) significantly enhances the response of the random system to weak perturbations with long periods.

2 Strongly damped Brownian particle

It is useful to roughly compare solutions of stochastic differential equations and motions of Brownian particles in double-well landscapes (Figure 3) in order to understand properties of their trajectories, see [14], [11]. As in the previous section, let us concentrate on a one-dimensional setting, remarking that we shall give a treatment that easily generalizes to the finite-dimensional setting. Due to Newton's law, the motion of a particle is governed by the impact of all forces acting on it. Let us denote F the sum of these forces, m the mass, x the space coordinate and v the velocity of the particle. Then

$$m\dot{v} = F.$$

Let us first assume the potential to be switched off. In their pioneering work at the turn of the twentieth century, Marian v. Smoluchowski and Paul Langevin introduced stochastic concepts to describe the Brownian particle motion by claiming that at time t

$$F(t) = -\gamma_0 v(t) + \sqrt{2k_B T\gamma_0}\dot{W}_t.$$

The first term results from friction γ_0 and is velocity dependent. An additional stochastic force represents random interactions between Brownian particles and their simple molecular random environment. The white noise $\dot{W}$ (formal derivative of the Wiener process) plays the crucial role. The diffusion coefficient (standard deviation of the random impact) is composed of Boltzmann's constant k_B, friction and environmental temperature T. It satisfies the condition of the fluctuation-dissipation theorem expressing the balance of energy loss due to friction and energy gain resulting from noise. The equation of motion becomes

$$\begin{cases} \dfrac{dx(t)}{dt} = v(t), \\ dv(t) = -\dfrac{\gamma_0}{m}v(t)dt + \dfrac{\sqrt{2k_B T\gamma_0}}{m}dW_t. \end{cases}$$

In the stationary regime, the stationary Ornstein-Uhlenbeck process provides its solution:

$$v(t) = v(0)\,e^{-\frac{\gamma_0}{m}t} + \frac{\sqrt{2k_B T\gamma_0}}{m}\int_0^t e^{-\frac{\gamma_0}{m}(t-s)}dW_s.$$

The ratio $\beta := \frac{\gamma_0}{m}$ determines the dynamic behavior. Let us focus on the over-damped situation with large friction and very small mass. Then for $t >> \frac{1}{\beta} = \tau$ (relaxation time), the first term in the expression for velocity can be neglected, while the stochastic integral represents a Gaussian process. By integrating, we obtain in the over-damped limit ($\beta \to \infty$) that v and thus x is Gaussian with almost constant mean

$$m(t) = x(0) + \frac{1 - e^{-\beta t}}{\beta}v(0) \approx x(0)$$

and covariance close to the covariance of white noise, see [13]:

$$\begin{aligned} K(s,t) &= \frac{2k_BT}{\gamma_0}\min(s,t) + \frac{k_BT}{\gamma_0\beta}\left(-2+2e^{-\beta t}+2e^{-\beta s}-e^{-\beta|t-s|}-e^{-\beta(t+s)}\right) \\ &\approx \frac{2k_BT}{\gamma_0}\min(s,t). \end{aligned}$$

Hence the time-dependent change of the velocity of the Brownian particle can be neglected, the velocity rapidly thermalizes ($\dot{v} \approx 0$), while the spactial coordinate remains far from equilibrium. In the so-called adiabatic transformation, the evolution of the particle's position is thus given by the transformed Langevin equation

$$dx(t) = \frac{\sqrt{2k_BT}}{\gamma_0}\,dW_t.$$

Let us next suppose that we have a Brownian particle in an external field of force (see Figure 3), generating a potential $U(t,x)$. This leads to the Langevin equation

$$\begin{cases} \dfrac{dx(t)}{dt} = v(t) \\ m\,dv(t) = -\gamma_0\,v(t)dt - \dfrac{\partial U}{\partial x}(t,x(t)) + \sqrt{2k_BT\gamma_0}\,dW_t. \end{cases}$$

In the over-damped limit, after relaxation time, the adiabatic elimination of the fast variables [8] leads to an equation similar to the one encountered in the previous section

$$dx(t) = -\frac{1}{\gamma_0}\frac{\partial U}{\partial x}(t,x(t)) + \frac{\sqrt{2k_BT}}{\gamma_0}\,dW_t.$$

In the particular case of some double-well potential $x \to U(t,x)$ with slow periodic variation, the following patterns of behavior of the solution trajectories will be experienced. If temperature is high, noise has a predominant influence on the motion, and the particle often crosses the barrier separating the two wells during one period. The behavior of the particle does not seem to be periodic but rather chaotic. If temperature is small, the particle stays for a very long time in the starting well, fluctuating weakly around the equilibrium position. It has too low energy to follow the periodic variation of the potential. So in this case too the trajectories do not look periodic. Between these two extreme situations, there exists a regime of noise intensities for which the energy transmitted by the noise is sufficient to cross the barrier almost twice per period. The parameters are then near to the resonance point and the motion exhibits periodic switching (Figure 4).

3 Transition criteria and quasi-deterministic motion

Studying stochastic resonance accordingly means looking for the range of regimes for which periodic behavior is enhanced and eventually optimal. The optimal relation between period T and noise intensity ε emerges in the small noise limit. To explain this, let us focus on the basic indicator for periodic transitions - the time the Brownian particle needs to exit from the starting well, say the left one. In the "frozen" case, that is if the time variation of the potential term is eliminated just by freezing it at some time s, the asymptotics of the exit time is derived from the classical large deviation theory of randomly perturbed dynamical systems, see Freidlin and Wentzell [4]. Let us assume that U is locally Lipschitz. We denote by D_l (resp. D_r) the domain corresponding to the left (resp. right) well and χ their common boundary. The law of the first exit time $\tau^\varepsilon_{D_l} = \inf\{t \geq 0,\ X^\varepsilon_t \notin D_l\}$ is described by some particular functional related to large deviation. For $t > 0$, we introduce the *action functional* on the space of continuous functions $\mathcal{C}([0,t])$ on $[0,t]$ by

$$S^s_t(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^t \left(\dot{\varphi}_u + \frac{\partial U}{\partial x}(s,\varphi_u)\right)^2 du, & \text{if } \varphi \text{ is abs. continuous,} \\ +\infty \ \ \text{otherwise,} \end{cases}$$

which is non-negative and vanishes on the set of solutions of the ordinary differential equation $\dot{x} = -\frac{\partial U}{\partial x}(s,x)$. Let x and $y \in \mathbb{R}$. In relation with the action functional, we define the quasipotential

$$V_s(x,y) = \inf\{S^s_t(\varphi):\ \varphi \in \mathcal{C}([0,t]),\ \varphi_0 = x,\ \varphi_t = y,\ t \geq 0\}.$$

It represents the minimal work the diffusion starting in x has to do in order to reach y. To switch wells, the Brownian particle starting in the left well's bottom x_l has to overcome the barrier. So we let

$$\overline{V}_s = \inf_{y \in \mathcal{X}} V_s(x_l, y).$$

This minimal work needed to exit from the left well can be computed explicitly, and is seen to equal to twice its depth. The asymptotic behavior of the exit time is expressed by

$$\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{E}[\tau^{\varepsilon}_{D_l}] = \overline{V}_s$$

and

$$\lim_{\varepsilon \to 0} \mathbb{P}_x(e^{\frac{\overline{V}_s - \delta}{\varepsilon}} < \tau^{\varepsilon}_{D_l} < e^{\frac{\overline{V}_s + \delta}{\varepsilon}}) = 1 \text{ for any } \delta > 0.$$

The prefactor for the exponential rate, derived by Freidlin and Wentzell [4] was first given by Eyring and Kramers and then by Bovier et al. [3]
Let us now assume that the left well is the deeper one at time s. If the Brownian particle has enough time to cross the barrier, i.e. if $T > e^{\frac{\overline{V}_s}{\epsilon}}$, then whatever the starting point is, Freidlin [6] proved that it should stay near x_l in the following sense

$$\Lambda(t \in [0,1]: \ |X^{\varepsilon}_{tT} - x_l| > \delta) \to 0$$

in probability as $\varepsilon \to 0$. Here Λ denotes Lebesgue measure on $\mathbb{R}$. If $T < e^{\frac{\overline{V}_s}{\epsilon}}$, the time left is not long enough for crossings: the particle stays in the starting well, near the stable equilibrium point:

$$\Lambda(t \in [0,1]: \ |X^{\varepsilon}_{tT} - (x_l 1_{\{x \in D_l\}} + x_r 1_{\{x \in D_r\}})| > \delta) \to 0.$$

This observation is at the basis of Freidlin's law of quasi-deterministic periodic motion discussed in the subsequent section. The lesson it teaches is this: to observe switching of the position to the energetically most favorable well, T should be larger than some critical level $e^{\frac{\lambda}{\epsilon}}$. Measuring time in exponential scales by μ through the equation $T^{\varepsilon} = e^{\frac{\mu}{\varepsilon}}$, the condition becomes $\mu > \lambda$.

4 Stochastic resonance for landscapes, frozen on half periods

This particular case has analytical advantages, since it allows to employ classical techniques of semigroup and operator theory. The situation is the following: let U be a double-well potential with minima $x_l = -1$ and $x_r = 1$ and a saddle point at the origin. We assume that $U(x) \to \infty$ as $|x| \to \infty$ and $U(-1) = -\frac{V}{2} = -\frac{\overline{V}_l}{2}$, $U(1) = -\frac{v}{2} = -\frac{\overline{V}_r}{2}$, $U(0) = 0$ and $0 < v < V$. We define the 1-periodic potential by $U(t,x) = U(t + \frac{1}{2}, -x)$. Hence on each half period the corresponding diffusion is time homogeneous. The critical level λ is then easily defined by $\lambda = v$, i.e. twice the depth of the shallow well. By letting

$$\phi(t) = \begin{cases} -1 \text{ for } t \in [k, k+\frac{1}{2}) \\ 1 \text{ for } t \in [k+\frac{1}{2}, k+1) \quad k = 0,1,2,... \end{cases}$$

the periodic function which describes the location of the global minimum of the potential, we get in the small noise limit

$$\Lambda(t \in [0,1]: \ |X^{\varepsilon}_{tT} - \phi(t)| > \delta) \to 0$$

in probability as $\varepsilon \to 0$. This result expresses Freidlin's law of quasi-deterministic motion: for large periods, the trajectories of the particle approach a periodic deterministic function. But the sense in which this notion measures periodicity does not take into account that for large periods short excursions to the wrong well may occur in an erratic way without counting much for Lebesgue measure of time. In fact, if the period is too large, that is $\mu > V$, the time available in one period permits the exit of the shallow well but that of the deep well too. So, even if the position of the particle is close to the bottom of the deep well, the number of transitions in one half-period becomes very large since starting in x_l the first time ξ the particle hits x_l after visiting the position x_r satisfies

$$\mathbb{E}(\xi) = e^{v/\varepsilon} + e^{V/\varepsilon} < T^{\varepsilon} = e^{\mu/\varepsilon}.$$

The motion of the particle appears more chaotical than periodic: noise intensity is too large compared to period length. We avoid this range of chaotic spontaneous transitions by defining the *resonance interval*

$I_R =]v, V[$, as the range of admissible energy parameters μ for randomly periodic behavior. In this regime, the trajectories possess periodicity properties. In these terms the *resonance point* describes the tuning rate $\mu_R \in I_R$ for which the stochastic response to weak external periodic forcing is optimal. To make sense, this point has to refer to some *measure of quality* for periodicity of random trajectories. In the huge physics literature concerning resonance, two families of criteria can be distinguished. The first one is based on invariant measures and spectral properties of the infinitesimal generator associated with the diffusion X^ε. Now, X^ε is not Markovian and consequently doesn't admit invariant measures. But by taking into account deterministic motion of time in the interval of periodicity and considering the process $Z_t = (t \bmod(T^\varepsilon), X_t)$, we obtain a Markov process with an invariant measure $\nu_t(x)dx$. In other words, the law of $X_t \sim \nu_t(x)dx$ and the law of $X_{t+T} \sim \nu_{t+T}(x)dx$, under this measure are the same for all $t \geq 0$. Let us present the most important ones:

- the *spectral power amplification* (SPA) which plays an eminent role in the physics literature describes the energy carried by the spectral component of the averaged trajectories of X^ε corresponding to the period:
$$M_{SPA}(\varepsilon, T) = \left| \int_0^1 \mathbb{E}_\nu[X^\varepsilon_{sT}] e^{2\pi i s} ds \right|^2.$$
- the *SPA-to-noise ratio*, giving the ratio of the amplitude of the response and the noise intensity, which is also related to the *signal to noise ratio*:
$$M_{SPN}(\varepsilon, T) = M_{SPA}(\varepsilon, T)/\varepsilon^2$$
- the total energy of the averaged trajectories
$$M_{EN}(\varepsilon, T) = \int_0^1 \left(\mathbb{E}_\nu[X_{sT}]\right)^2 ds.$$

The second family of criteria is more probabilistic. It refers to quality measures based on transition times between the domains of attraction of the local minima, residence times distributions measuring the time spent in one well between two transitions, or interspike times. This family is certainly less popular in the physics community.

However, measures related to invariant measures may suffer from robustness deficiency [10]. To explain what we mean by robustness, let us introduce a model reduction first discussed by McNamara, Wiesenfeld [12]. Instead of studying the diffusion X^ε in the double-well landscape, they introduce a two state Markov chain Y^ε (Figure 5) the dynamics of which just takes account of the domain of attraction the diffusion is in, and therefore with state space $\{-1, 1\}$. A reasonable choice of the infinitesimal generator should retain the dynamics of the diffusion's transitions characterized by Kramers' rate. We may take

$$Q(t) = \begin{pmatrix} -\varphi & \varphi \\ \psi & -\psi \end{pmatrix}, \quad 0 \leq t \leq \frac{T}{2}, \quad Q(t) = \begin{pmatrix} -\psi & \psi \\ \varphi & -\varphi \end{pmatrix}, \quad \frac{T}{2} \leq t < T,$$

periodically continued on $\mathbb{R}_+$. Here $\varphi = pe^{-V/\varepsilon}$ and $\psi = qe^{-v/\varepsilon}$. The prefactors of sub-exponential order are beyond the scope of large deviation theory. They are related to the curvature of the potential in the minima and the saddle point of the landscape and given by

$$p = \frac{1}{2\pi} \sqrt{U''(-1)|U''(0)|}, \qquad q = \frac{1}{2\pi} \sqrt{U''(1)|U''(0)|}.$$

On the intervals $[kT/2, (k+1)T/2[$, $k \geq 0$, the Markov chain Y^ε is time-homogeneous and its transition probabilities can be expressed in terms of φ and ψ. For instance, the probability with which the chain jumps from state -1 to state $+1$ in the time window $[t, t+h]$ equals $\varphi h + o(h)$, if this time interval is contained in $[kT/2, (k+1)T/2[$ for some even k. The stationary measure of the Markov chain denoted by ν can be explicitly calculated, and so do the classical quality measures based on the spectral notions. For instance, the spectral power amplification coefficient equals

$$M_{SPA}(\varepsilon, T) = \left| \int_0^1 \mathbb{E}_\nu[Y^\varepsilon_{st}] e^{2\pi i s} ds \right|^2 = \frac{4}{\pi^2} \frac{T^2(\varphi - \psi)^2}{(\varphi + \psi)^2 T^2 + \pi^2}.$$

This simple expression admits asymptotically a unique maximum which exhibits the resonance point:

$$T^\varepsilon_{opt} = \frac{\pi}{\sqrt{2pq}} \sqrt{\frac{v}{V - v}}\, e^{\frac{V+v}{2\varepsilon}} \left\{1 + \mathcal{O}(e^{-\frac{V-v}{\varepsilon}})\right\}.$$

The optimal period is then exponentially large - as was suggested by large deviation theory - and the growth rate is the sum of the two wells' depths. The simple Markov chain model is popular since the usual physical quantities are easy computable and since it is believed to mimic the dynamics of a Brownian particle in the corresponding double-well landscape. However, the models are not as similar as expected [5]. Indeed, in a reasonably large time window around the resonance point for Y^ε, the tuning picture of the spectral power amplification for the diffusion is different. Under weak regularity conditions on the potential, it exhibits strict monotonicity in the window. Hence optimal tuning points for diffusion and Markov chain differ essentially. In other words, the diffusion's SPA tuning behavior is not robust for passage to the reduced model. This strange deficiency is difficult to explain. The main reason of this subtle effect appears to be that the diffusive nature of the Brownian particle is neglected in the reduced model. In order to point out this feature, we may compute the SPA coefficient of $g(X^\varepsilon)$ where g is a particular function designed to cut out the small fluctuations of the diffusion in the neighborhood of the bottoms of the wells, by identifying all states there. So $g(x) = -1$ (resp. 1) in some neighborhood of -1 (resp. 1) and otherwise g is the identity. This results in

$$\widetilde{M}_{SPA}(\varepsilon, T) = \left| \int_0^1 \mathbb{E}_\nu g(X^\varepsilon_{sT})\, e^{2\pi i s} ds \right|^2.$$

In the small noise limit this quality function admits a local maximum close to the resonance point of the reduced model: the growth rate of T^ε_{opt} is also given by the sum of the wells' depths. So the lack of robustness seems to be due to the small fluctuations of the particle in the wells' bottoms. In any case, this clearly calls for other quality measures to be used to transfer properties of the reduced model to the original one. Our discussion indicates that due to their emphasis on the pure transition dynamics, the second family of quality measures should be used. For these notions there is no need to restrict to landscapes frozen in time independent potential states on half period intervals.

5 Stochastic resonance for continuously varying landscapes

From now on the potential $U(t, x)$ is supposed to be continuously varying in (t, x). For simplicity its local minima are assumed to be located at ± 1, and its only saddle point at 0, independently of time. So the only meta-stable states on the whole time axis are ± 1. Let us denote by $\Delta_-(t)$ (resp. $\Delta_+(t)$) the depth of the left (resp. right) well at time t. Together with U, these functions are continuous and 1-periodic. Assume that they are strictly monotonous between their global extrema. Let us now come back to the motion of a Brownian particle in this landscape. The exit time law by Eyring-Kramers-Freidlin entails that trajectories get close to the global minimum, if the period is large enough. Stated as before in exponential rates $T = e^{\mu/\varepsilon}$, with $\mu \geq \max_{i=\pm} \sup_{t\geq 0} 2\Delta_\pm(t)$, that is, μ exceeds the maximal work needed to cross the barrier, the particle often switches between the two wells and should stay close to the deepest position in the landscape. This position being described by the function $\phi(t) = 2\, 1_{\{\Delta_+(t)>\Delta_-(t)\}} - 1$, we get in the small noise limit

$$\Lambda(t \in [0, 1] : \ |X^\varepsilon_{tT} - \phi(t)| > \delta) \to 0.$$

in probability. But on these long time scales, many short excursions to the wrong well are observed, and trajectories look chaotical instead of periodic. So we have to look at smaller periods even at the cost that the particle may not stay close to the global minimum. Let us study the transition dynamics. Assume that the starting point is -1 corresponding to the bottom of the deep well. If the depth of the well is always larger than $\mu = \varepsilon \log T^\varepsilon$, the particle has too little time during one period to climb the barrier and should stay in the starting well. If on the contrary the minimal work to leave the starting well, given by $2\Delta_-(s)$, becomes smaller than μ at some time s, then the transition can and will happen. More formally, for $\mu \in]\inf_{t\geq 0} 2\Delta_-(t), \sup_{t\geq 0} 2\Delta_-(t)[$ we define (Figure 6)

$$a^-_\mu(s) = \inf\{t \geq s : 2\Delta_-(t) \leq \mu\}.$$

The first transition time from -1 to 1 denoted τ_+ has the following asymptotic behavior as $\varepsilon \to 0$: $\tau_+/T^\varepsilon \to a^-_\mu(0)$. At the second transition the particle returns to the starting well. If a^+_μ is defined analogously with respect to the depth function Δ_+, this transition will occur near the deterministic time $a^+_\mu(a^-_\mu(s))T^\varepsilon$. In order to observe periodicity, and to exclude chaoticity from all parts of its trajectories, the particle has to stay for some time in the other well before returning. This will happen under the assumption $2\Delta_+(a_\mu(0)) > \mu$, that is, the right well is the deep one at transition time. In fact we can define the *resonance interval* I_R (Figure 7), as the set of all scales μ for which trajectories exhibit periodicity in the small noise limit, by

$$I_R =]\max_{i=\pm}\inf_{t\geq 0} 2\Delta_i(t), \inf_{t\geq 0}\max_{i=\pm} 2\Delta_i(t)[.$$

On this interval they get close to deterministic periodic ones. Again, periodicity is quantified by a quality measure, to be maximized in order to obtain resonance as the best possible response to periodic forcing. One interesting measure is based on the probability that random transitions happen in some small time window around a deterministic time, in the small noise limit [9]. Formally, for $h > 0$, the measure gives

$$M_h(\varepsilon, T) = \min_{i=\pm} \mathbb{P}_i(\tau_{\mp}/T^{\varepsilon} \in [a_\mu^i - h, a_\mu^i + h]),$$

where $\mathbb{P}_i$ is the law of the diffusion starting in i. In the small noise limit, this quality measure tends to 1, and optimal tuning can be related to the exponential rate at which this happens. This is due to the following large deviations principle:

$$\lim_{\varepsilon\to 0} \varepsilon \log(1 - M_h(\varepsilon, T)) = \max_{i=\pm}\{\mu - 2\Delta_i(a_\mu^i - h)\}$$

for $\mu \in I_R$, with uniform convergence on each compact subset of I_R. The result is established using classical large deviations techniques applied to locally time homogeneous approximations of the diffusion. Maximizing the transition probability in the time window position means minimizing the default rate obtained by the large deviations principle. This can be easily achieved. In fact, if the windows length $2h$ is small, then $\mu - 2\Delta_i(a_\mu^i - h) \approx 2h\Delta_i'(a_\mu^i)$, since $2\Delta_i(a_\mu^i) = \mu$ by definition. The value $\Delta_i'(a_\mu^i)$ is negative, so we have to find the position where its absolute value is maximal. In this position the depth of the starting well has the most rapid drop under the level μ, characterizing the link between the noise intensity and the period. So the transition time is best concentrated around it.
It is clear that a good candidate for the resonance point is given by the eventually existing limit of the global minimizer $\mu_R(h)$ as the window length h tends to 0. This limit is therefore called *resonance point* of the diffusion with time periodic landscape U. Let us note that for sinusoidal depth functions $\Delta_-(t) = \frac{V+v}{4} + \frac{V-v}{4}\cos(2\pi t)$ and $\Delta_+(t) = \Delta_-(t+\pi)$ the optimal tuning is given by $T^\varepsilon = \exp\frac{\mu_R}{\varepsilon}$ with $\mu_R = \frac{v+V}{2}$. This optimal rate is equivalent to the optimal rate given by the SPA coefficient of the reduced dynamics' Markov chain in the preceding section.

The big advantage of the quality measure M_h is its robustness. Indeed, consider the reduced model consisting of a two-state Markov chain with infinitesimal generator

$$Q(t) = \begin{pmatrix} -\varphi(t) & \varphi(t) \\ \psi(t) & -\psi(t) \end{pmatrix},$$

where $\varphi(t) = \exp -\frac{2\Delta_-(t/T)}{\varepsilon}$ and $\psi(t) = \exp -\frac{2\Delta_+(t/T)}{\varepsilon}$. The law of transition times of this Markov chain is readily computed from Laplace transforms. Normalized by T^ε it converges to a_μ^i. This calculation even reveals a rigorous underlying pattern for the second and higher order transition times interpreting the interspike distributions of the physics literature. The dynamics of diffusion and Markov chain are similar. Resonance points provided by M_h for the diffusion and its analogue for the Markov chain agree.

6 Related notions: synchronization

In the preceding sections, we interpreted stochastic resonance as optimal response of a randomly perturbed dynamical system to weak periodic forcing, in the spirit of the physics literature, see Gammaitoni et al. [7]. Our crucial assumption concerned the barrier heights a Brownian particle has to overcome in the potential landscape of the dynamical system: it is uniformly lower bounded in time. Measures for the quality of tuning were based on essentially two concepts: one concerning spectral criteria, with the spectral power amplification as most prominent member, the other one concerning the pure transitions dynamics between the domains of attraction of the local minima. A number of different criteria can be used to create an optimal tuning between the intensity of the noise perturbation and the large period of the dynamical system. The relations have to be of an exponential type $T = \exp\frac{\mu}{\varepsilon}$, since the Brownian particle needs exponentially long times to cross the barrier separating the wells according to the Eyring-Kramers-Freidlin transition law. Our barrier height assumption seems natural in many situations, but

can fail in others. If it becomes small periodically, and eventually scales with the noise intensity parameter, the Brownian particle doesn't need to wait an exponentially long time to climb it. So periodicity obtains for essentially smaller time scales. In this setting, the slowness of periodic forcing may also be assumed to be essentially sub-exponential in the noise intensity.
If it is fast enough to allow for substantial changes before large deviation effects can take over, we are in the situation of Berglund and Gentz [2]. They in fact consider the case in which the barrier between the wells becomes low twice per period, to the effect of modulating periodically a bifurcation parameter: at time zero the right-hand well becomes almost flat and at the same time the bottom of the well and the saddle approach each other; half a period later, a spatially symmetric scenario is encountered. In this situation, there is a threshold value for the noise intensity under which transitions become unlikely. Above this threshold, the trajectories typically contain two transitions per period. Results are formulated in terms of concentration properties for random trajectories. The intuitive picture is this: with overwhelming probability, sample paths will be concentrated in space–time sets scaling with the small parameters of the problem. In higher dimensions, these sets may be given by adiabatic or center manifolds of the deterministic system, which allow model reduction of higher-dimensional systems to lower-dimensional ones. Asymptotic results hold for any choice of the small parameters in a whole parameter region. A passage to the small noise limit as for optimal tuning in the preceding sections is not needed.
Related problems studied by Berglund and Gentz in the multidimensional case concern the noise-induced passage through periodic orbits, where unexpected phenomena arise. Here, as opposed to the classical Freidlin-Wentzell theory, the distribution of first-exit points depends non-trivally on the noise intensity. Again aiming at results valid for small but non-vanishing parameters in sub-exponential scale ranges, they investigate the density of first-passage times in a large regime of parameter values, and obtain insight into the transition from the stochastic resonance regime into the synchronisation regime.

See also

Spectral theory for linear operators, stochastic differential equations, resonances, magnetic resonance imaging, dynamical systems in mathematical physics.

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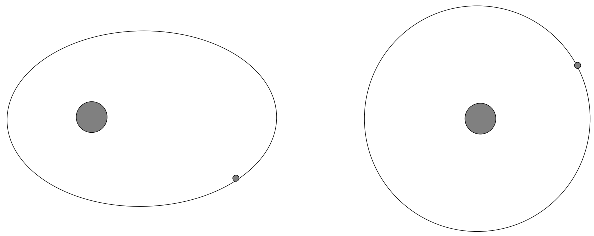

Figure 1: Modulation of the orbital eccentricity

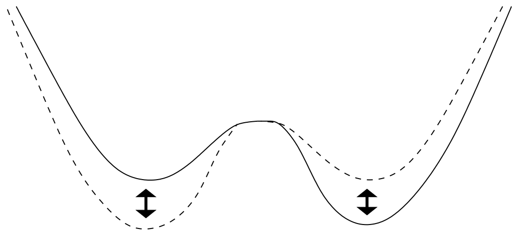

Figure 2: Deep and shallow wells switching periodically

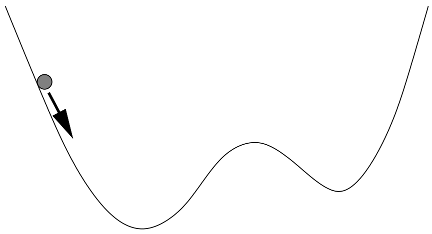

Figure 3: Brownian particle in a double-well landscape

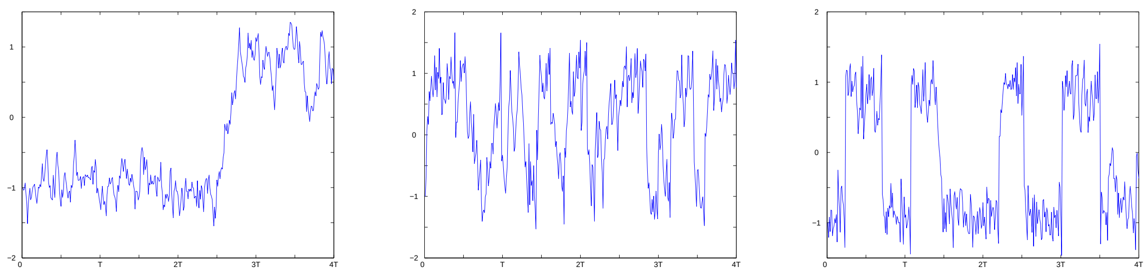

Figure 4: Resonance pictures for diffusions

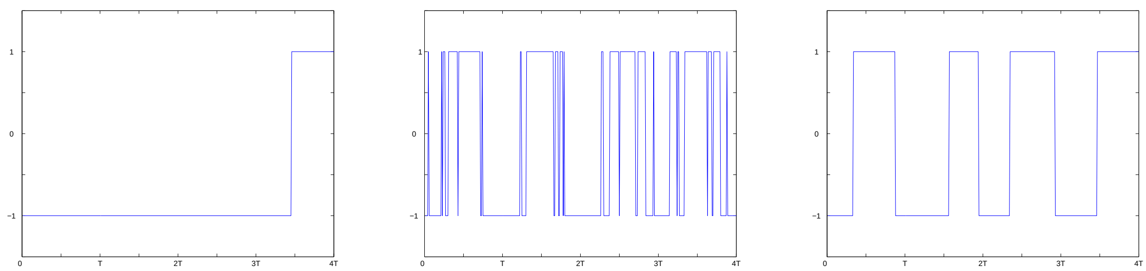

Figure 5: Resonance pictures for Markov chain.

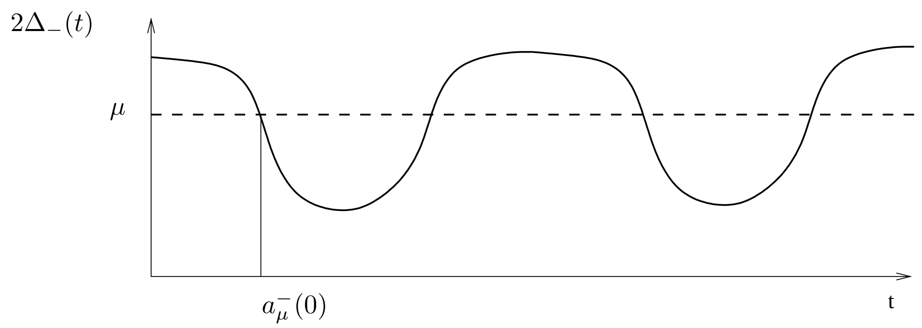

Figure 6: Definition of a_μ^-

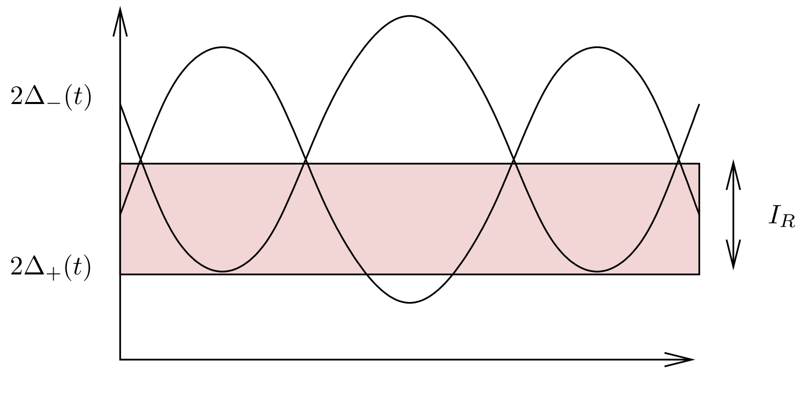

Figure 7: Resonance interval