A singular large deviations phenomenon

Mihai Gradinaru, Samuel Herrmann and Bernard Roynette

Institut de Mathématiques Elie Cartan, Université Henri Poincaré,
B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France
{*Mihai.Gradinaru,Samuel.Herrmann,Bernard.Roynette*}*@iecn.u-nancy.fr*

Abstract: Consider $\{X_t^\varepsilon : t \geq 0\}$ ($\varepsilon > 0$), the solution starting from 0 of a stochastic differential equation, which is a small Brownian perturbation of the one-dimensional ordinary differential equation $x_t' = \mathrm{sgn}(x_t)|x_t|^\gamma$ ($0 < \gamma < 1$). Denote by $p_t^\varepsilon(x)$ the density of X_t^ε. We study the exponential decay of the density as $\varepsilon \downarrow 0$. We prove that, for the points (t, x) lying between the extremal solutions of the ordinary differential equation, the rate of the convergence is different from the rate of convergence in large deviations theory (although respected for the points (t, x) which does not lie between the extremals). Proofs are based on probabilistic (large deviations theory) and analytic (viscosity solutions for Hamilton-Jacobi equations) tools.

1991 Mathematics Subject Classifications: 60F10, 60H10, 34F05, 35G20, 35G30, 70H20, 60J65

Keywords: large deviations, small random perturbation, Brownian bridge, viscosity solution, Hamilton-Jacobi equation

INTRODUCTION

Let $0 < T < \infty$, $\{B_t : t \geq 0\}$ an one-dimensional Brownian motion, and consider the stochastic differential equation on $[0, T]$:

$$\begin{cases} dX_t^\varepsilon = \varepsilon dB_t + \mathrm{b}(X_t^\varepsilon)dt \\ X_0^\varepsilon = x_0. \end{cases}$$

Let us denote by P_ε the law of the process $X_\cdot^\varepsilon$. It is classical that the family $\{P_\varepsilon : \varepsilon > 0\}$ is weakly relatively compact and, as ε tends to zero, every cluster value P has its support contained in the set of paths x which are solutions of the dynamical system

$$\begin{cases} x'(t) = \mathrm{b}(x(t)) \\ x(0) = x_0. \end{cases} \tag{1}$$

If (1) has an unique solution (for instance, if b is a Lipschitz function), then by the large deviations theory, it is known that P_ε is exponentially tight and therefore P_ε converges to P exponentially fast, as ε tends to zero.

If (1) has more than one solution, in [B-B] it is proved that, under suitable conditions, there is just one limit value in law, concentrated on at most two paths: the extremal solutions of (1) (see Figure 1).

The aim of this paper is to study the precise convergence of P_ε towards P in the following case: take $0 < \gamma < 1$ and let P_ε be the law of the solution of the stochastic differential equation:

$$\begin{cases} dX_t^\varepsilon = \varepsilon\, dB_t + \mathrm{sgn}(X_t^\varepsilon)|X_t^\varepsilon|^\gamma dt \\ X_0^\varepsilon = 0. \end{cases} \tag{2}$$

We can see this equation as a small random perturbation of the dynamical system:

$$\begin{cases} x_t' = \mathrm{sgn}(x_t)|x_t|^\gamma \\ x_0 = 0. \end{cases} \tag{3}$$

Let us denote by $p_t^\varepsilon(\cdot)$ the density of X_t^ε with respect to the Lebesgue measure. We observe that if $|x| \neq (t(1-\gamma))^{1/1-\gamma}$, *i.e.* if (t,x) does not belong to the graph of one of the extremal solutions of problem (3), then the density tends to zero, justifying the results in [B-B].

Let us note that, if we assume that $\gamma = 0$, the calculation is explicit and the behaviour is the same.

Let us describe our main results. According to the position of the point (t,x), we emphasise two kinds of rate.

If the point (t,x) is such that $|x| > \{t(1-\gamma)\}^{1/1-\gamma}$, there exists a positive function k_t such that:

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x) = -k_t(|x|).$$

This means that the density has an exponential decay with rate ε^2, as in large deviations theory. The rate is the same as in the case when the dynamical system has an unique solution. For instance, if the drift b is a Lipschitz function the rate agrees to the rate in Freidlin-Wentzell theorem for random perturbations of the dynamical systems (see, [De-S], p. 31).

If the point (t,x) lies in the domain between the two extremals, that is, if $|x| \le \{t(1-\gamma)\}^{1/1-\gamma}$, then the density has an exponential decay with a different rate, namely $\varepsilon^{2(1-\gamma)/(1+\gamma)}$. Precisely, we show that, for such points (t,x):

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{2(1-\gamma)}{(1+\gamma)}} \ln p_t^\varepsilon(x) = \lambda_1 \left(\frac{x^{1-\gamma}}{1-\gamma} - t\right).$$

Here λ_1 is the first positive eigenvalue of the Schrödinger operator on $L^2(\mathbb{R})$:

$$-\frac{1}{2}\frac{d^2}{dx^2} + \frac{\gamma}{2|x|^{1-\gamma}} + \frac{|x|^{2\gamma}}{2}.$$

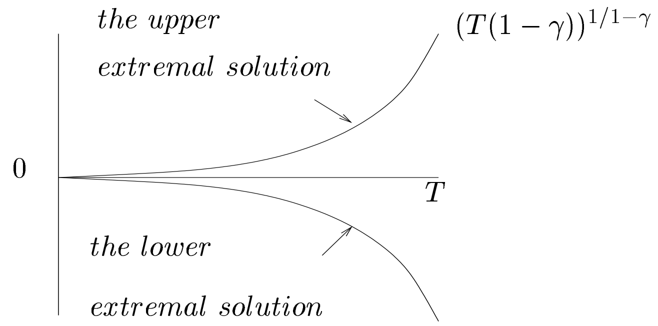

Figure 1: Solutions of the dynamical system

The plan of the paper is as follows. In the first section we recall some existence results for stochastic and ordinary equations and also the results of [B-B], for the drift $\mathrm{b}(x) := \mathrm{sgn}(x)\,|x|^\gamma$, $0 < \gamma < 1$. Also we make the study for the case of the exponent $\gamma = 0$. In §2 we give some representations of the density p_t^ε. In particular, we give an expansion in terms of eigenvalues and eigenfunctions of the Schrödinger operator. This was already studied by Kac [K] for continuous potentials and we adapt this result to our situation. Section §3 is devoted to the convergence of the density in logarithmic scale with rate ε^2. We compute the limit for the points (t,x) which does not lie between the extremals (see Theorem 1) and we give an upper bound for the other points. In the last two sections we treat the convergence of the density in logarithmic scale with the rate $\varepsilon^{2(1-\gamma)/(1+\gamma)}$, for the points (t,x) lying between the extremals. An upper bound is obtained in §4, for the exponent $\gamma = 1/2$, using large deviations arguments (see Theorem 2). In particular, we compute the behaviour of

$$\lim_{\varepsilon \to 0} \varepsilon^{2/3} \ln \mathbb{E}\left[\exp -\frac{1}{\varepsilon}\int_0^t |b_s|\, ds\right],\ 0 < t < 1,$$

where $\{b_s : s \in [0,1]\}$ is the standard Brownian bridge (see Proposition 7). The precise limit is obtained in Theorem 3 by the study, developed in §5, of the viscosity solution of a Hamilton-Jacobi equation. Although the ideas are inspired by [B], there are several new difficulties, since, for example, b is not a Lipschitz function.

1. EXISTENCE RESULTS

In this section we recall some existence results for the stochastic differential equation (2), for the ordinary differential equation and the convergence result of [B-B].

Proposition 1 *There exists a unique strong solution of* (2)*.*

Proof: The drift of the diffusion $\mathrm{b}(x) = \mathrm{sgn}(x)|x|^\gamma$ is a continuous function and satisfies

$$\int_y^{y+\gamma} \mathrm{sgn}(x)|x|^\gamma dx < \infty,\ \forall y \in \mathbb{R},$$

so by a classical result (see, for instance, [K-S] Theorem 5.15, p. 341) we get the local existence and the uniqueness in law.
Moreover, the local solution is in fact a global solution. Indeed, the explosion time of the diffusion is a.s. equal to infinity (see, for instance [I-W], Theorem 2.4, p. 177): since $0 < \gamma < 1$, there exists a positive constant K, such that

$$\varepsilon^2 + \mathrm{b}(x)^2 = \varepsilon^2 + |x|^{2\gamma} \le K(1 + |x|^2).$$

Let us prove now the pathwise uniqueness. Let X^1, X^2 be two solutions of (2):

$$X_t^1 - X_t^2 = \int_0^t (\mathrm{b}(X_s^1) - \mathrm{b}(X_s^2))ds.$$

The local time at level zero of the process $X^1 - X^2$ is given by

$$L_t^0(X^1 - X^2) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}(X_s^1 - X_s^2)\, d < X^1 - X^2, X^1 - X^2 >_s = 0,$$

since $X^1 - X^2$ is a finite variation process.
Combining this equality with the uniqueness in law, we deduce that the two paths X^1, X^2 starting from zero are the same a.s. (see also, [R-Y] Proposition 3.2, p. 370). **QED**

We study now the dynamical system (3) and the behaviour of the law P_ε of the process $X_\cdot^\varepsilon$, as $\varepsilon \to 0$:

Proposition 2 *The equation* (3) *admits an infinity of solutions:*

$$\left\{c_\gamma(t-\lambda)_+^{1/1-\gamma}, \lambda \geq 0 \quad -c_\gamma(t-\lambda)_+^{1/1-\gamma}, \lambda \geq 0\right\}$$

where c_γ is a constant. Let us denote by

$$\rho_{1,2}(t) = \pm\{(1-\gamma)t\}^{1/1-\gamma}$$

the extremal solutions of the dynamical system. Then P_ε tends to $\frac{1}{2}\delta_{\rho_1} + \frac{1}{2}\delta_{\rho_2}$, as $\varepsilon \to 0$.

Proof: We shall use the results in [B-B]. Since, for $r > 0$, $\int_0^r 1/\mathrm{b}(y)dy < \infty$, we can apply Proposition 2.1 in [B-B], p. 280, and we get the existence of an infinity of solutions.
By Theorem 5.2 in [B-B], p. 291: if P is any cluster value of $\{P_\varepsilon\}$, as $\varepsilon \to 0$,then, for all t, $P|_{\mathcal{F}_t}$, is concentrated on the extremal solutions ρ_1 and ρ_2:

$$P_{|\mathcal{F}_t} = \frac{1}{2}\delta_{\rho_1} + \frac{1}{2}\delta_{\rho_2}.$$

Here $\mathcal{F}_t$ is the Brownian filtration on $[0,t]$. **QED**

Remark 1 Let us note that in the case $\gamma = 0$ the calculation is explicit, we compute the density and we show that the diffusion tends towards the extremal solutions (in a generalized sense, namely a.e. differentiable) of the following differential equation:

$$\begin{cases} x'_t = \mathrm{sgn}(x_t) \\ x_0 = 0, \end{cases}$$

which are $\rho_{1,2}(t) = \pm t$. We represent the convergence by arrows in Figure 2.

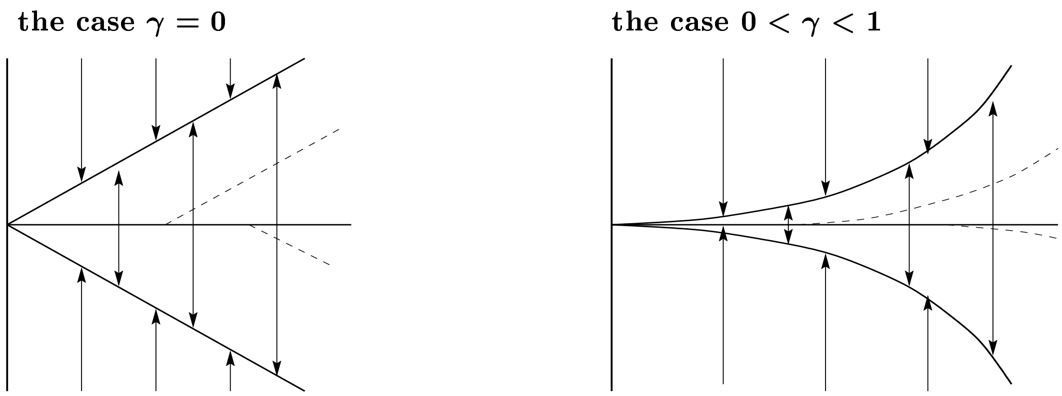

Figure 2: Convergence of the diffusion

In this particular case, the diffusion $X^\varepsilon_\cdot$ is solution of the following stochastic differential equation:

$$\begin{cases} dX^\varepsilon_t = \varepsilon dB_t + \mathrm{sgn}(X^\varepsilon_t)dt \\ X^\varepsilon_0 = 0, \end{cases} \qquad (2')$$

and we can compute the density $p^\varepsilon_t(x)$ of X^ε_t with respect to the Lebesgue measure:

Proposition 3 *Let us denote $\varphi(x) = \int_x^\infty e^{-y^2/2}dy$. Then,*

$$p^\varepsilon_t(x) = \frac{1}{\varepsilon\sqrt{2\pi t}}\exp-\left\{\frac{(|x|-t)^2}{2\varepsilon^2 t}\right\} - \frac{1}{\varepsilon^2\sqrt{2\pi}}\varphi\left(\frac{|x|}{\varepsilon\sqrt{t}} + \frac{\sqrt{t}}{\varepsilon}\right)\exp\frac{2|x|}{\varepsilon^2}. \qquad (4)$$

Moreover,

$$\lim_{\varepsilon\to 0}\varepsilon^2 \ln p_t^\varepsilon(x) = -\frac{(|x|-t)^2}{2t}$$

and

$$\lim_{\varepsilon\to 0}\varepsilon^2 \ln \mathbb{P}(||X_t^\varepsilon|-t|\geq\delta) = -\frac{\delta^2}{2t}.$$

Proof: Let us consider the martingale (see, for instance, [K-S] p. 352 and also [B-R-V] for other applications of a similar calculation)

$$M_t = \exp\left\{\frac{1}{\varepsilon}\int_0^t \operatorname{sgn}(B_s^\varepsilon)dB_s - \frac{t}{2\varepsilon^2}\right\} = \exp\left\{\frac{|B_t|}{\varepsilon} - \frac{2L_t}{\varepsilon} - \frac{t}{2\varepsilon^2}\right\},$$

as we can see by Tanaka's formula (see, for instance, [K-S] Proposition 6.8, p. 205). Here L_t is the Brownian local time at level 0.
Since the equation (2') can be written as

$$\frac{X_t^\varepsilon}{\varepsilon} = B_t + \frac{1}{\varepsilon}\int_0^t \operatorname{sgn}(X_s^\varepsilon)ds,$$

using Girsanov formula we get, for every Borel function f,

$$\mathbb{E}\left[f\left(\frac{X_t^\varepsilon}{\varepsilon}\right)\right] = \mathbb{E}[f(B_t)M_t].$$

Take f an even function. Then

$$\mathbb{E}[f(X_t^\varepsilon)] = \mathbb{E}\left[f(\varepsilon|B_t|)\exp\left\{\frac{|B_t|}{\varepsilon} - \frac{2L_t}{\varepsilon} - \frac{t}{2\varepsilon^2}\right\}\right].$$

Moreover, since $(|B_t|, 2L_t)$ has the same law as $(S_t - B_t, S_t)$, where $S_t = \sup_{0\leq s\leq t} B_s$, we get

$$\mathbb{E}[f(X_t^\varepsilon)] = \mathbb{E}\left[f(\varepsilon(S_t - B_t))\exp\left\{-\frac{B_t}{\varepsilon} - \frac{t}{2\varepsilon^2}\right\}\right]$$

where the law of (B_t, S_t) is well known (see, for instance, [K-S] Proposition 8.1, p. 95):

$$\mathbb{P}(B_t\in da, S_t\in db) = \frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(2b-a)^2}{2t}\right\}dadb, \ \text{ for } \ a\leq b \quad , \ b\geq 0.$$

Hence

$$\mathbb{E}[f(X_t^\varepsilon)] = \int_0^\infty\int_{-\infty}^b \frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(2b-a)^2}{2t} - \frac{a}{\varepsilon} - \frac{t}{2\varepsilon^2}\right\} f(\varepsilon(b-a))dadb.$$

We make the change of variables $x := \varepsilon(2b-a)$ and $y := \varepsilon(b-a)$ and we obtain

$$\begin{aligned}
\mathbb{E}[f(X_t^\varepsilon)] &= \frac{2}{\varepsilon^3\sqrt{2\pi t^3}}\int_0^\infty\int_y^\infty x\exp\left\{-\frac{x^2}{2\varepsilon^2 t} + \frac{2y}{\varepsilon^2} - \frac{x}{\varepsilon^2} - \frac{t}{2\varepsilon^2}\right\} f(y)dxdy \\
&= \frac{2}{\varepsilon\sqrt{2\pi t}}\int_0^\infty f(y)\exp\left\{\frac{2y}{\varepsilon^2} - \frac{(y+t)^2}{2\varepsilon^2 t}\right\}dy \\
&- \frac{2}{\varepsilon^2\sqrt{2\pi}}\int_0^\infty dy\, f(y)\exp\frac{2y}{\varepsilon^2}\int_{(y+t)/(\varepsilon\sqrt{t})}^\infty \exp-\frac{v^2}{2}dv
\end{aligned}$$

From this equality we get the expression of the density (4).
Moreover, using the Laplace method we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^2\ln p_t^\varepsilon(x) = \left(2x - \frac{(x+t)^2}{2t}\right).$$

Let us now compute $\mathbb{P}(||X_t^\varepsilon| - t| \geq \delta)$. By (4), we get

$$\frac{2}{\varepsilon\sqrt{2\pi t}}\int_\delta^\infty (1-\frac{t}{x+t})\exp-\frac{(x-t)^2}{2\varepsilon^2 t}dx \leq \mathbb{P}(||X_t^\varepsilon|-t|\geq\delta) \leq \frac{1}{\varepsilon\sqrt{2\pi t}}\int_{||x|-t|\geq\delta}\exp-\frac{(|x|-t)^2}{2\varepsilon^2 t}dx.$$

By the Laplace method we get

$$\lim_{\varepsilon\to 0}\varepsilon^2\ln\mathbb{P}(||X_t^\varepsilon| - t|\geq\delta) = -\inf_{x\in\mathbb{R}\ ;\ ||x|-t|\geq\delta}\frac{(|x|-t)^2}{2t} = -\frac{\delta^2}{2t}.$$

QED

2. SOME REPRESENTATIONS OF THE DENSITY

In this section we shall describe some useful representations of the density of X_t^ε solution of the equation (2).

Proposition 4 *For $t>0$, $\varepsilon>0$, $x\in\mathbb{R}$:*

$$p_t^\varepsilon(x) = \tfrac{1}{\varepsilon\sqrt{2\pi t}}\exp\left\{\tfrac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \tfrac{x^2}{2\varepsilon^2 t}\right\}$$
$$\times\mathbb{E}\left[\exp\left\{-\frac{\gamma t}{2}\int_0^1 |xs+\varepsilon\sqrt{t}b_s|^{\gamma-1}ds - \frac{t}{2\varepsilon^2}\int_0^1|xs+\varepsilon\sqrt{t}b_s|^{2\gamma}ds\right\}\right], \tag{5}$$

where $\{b_t : t\in[0,1]\}$ is the standard Brownian bridge.

Proof: The equation (2) can be written as:

$$\frac{X_t^\varepsilon}{\varepsilon} = B_t + \frac{1}{\varepsilon^{1-\gamma}}\int_0^t \operatorname{sgn}(X_s^\varepsilon)\left|\frac{X_s^\varepsilon}{\varepsilon}\right|^\gamma ds,\ \text{for } t\geq 0.$$

Since the explosion time of the diffusion is a.s. infinite, the process

$$M_t = \exp\left\{\frac{1}{\varepsilon^{1-\gamma}}\int_0^t \operatorname{sgn}(B_s)|B_s|^\gamma dB_s - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t |B_s|^{2\gamma}ds\right\}$$

is a martingale (see, for instance, [K-S] Exercise 5.3.8, p. 352). Applying Girsanov theorem, we obtain:

$$\mathbb{E}\left[f\left(\frac{X_t^\varepsilon}{\varepsilon}\right)\right] = \mathbb{E}\left[f(B_t)\exp\left\{\frac{1}{\varepsilon^{1-\gamma}}\int_0^t\operatorname{sgn}(B_s)|B_s|^\gamma dB_s - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t|B_s|^{2\gamma}ds\right\}\right]. \tag{6}$$

Furthermore, by Ito's formula:

$$\int_0^t \operatorname{sgn}(B_s)|B_s|^\gamma dB_s = -\frac{\gamma}{2}\int_0^t |B_s|^{\gamma-1}ds + \frac{|B_t|^{\gamma+1}}{\gamma+1}$$

and replacing $f\left(\frac{\cdot}{\varepsilon}\right)$ by f in (6), we can write:

$$\mathbb{E}[f(X_t^\varepsilon)] = \mathbb{E}\left[f(\varepsilon B_t)\exp\left\{\frac{|B_t|^{\gamma+1}}{(\gamma+1)\varepsilon^{1-\gamma}} - \frac{\gamma}{2\varepsilon^{1-\gamma}}\int_0^t |B_s|^{\gamma-1}ds - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t |B_s|^{2\gamma}ds\right\}\right] \quad (7)$$

By the scaling property of the Brownian motion the right hand side of (7) can be written as:

$$\mathbb{E}\left[f(\varepsilon\sqrt{t}B_1)\exp\left\{\frac{t^{(\gamma+1)/2}}{(\gamma+1)\varepsilon^{1-\gamma}}|B_1|^{\gamma+1} - \frac{\gamma t^{(\gamma+1)/2}}{2\varepsilon^{1-\gamma}}\int_0^1 |B_s|^{\gamma-1}ds - \frac{t^{\gamma+1}}{2\varepsilon^{2-2\gamma}}\int_0^1 |B_s|^{2\gamma}ds\right\}\right].$$

Let us decompose the Brownian motion as follows:

$$B_t = g\,t + b_t,$$

where g is a standard Gaussian random variable independent of the Brownian bridge b. Therefore,

$$\mathbb{E}[f(X_t^\varepsilon)] = \int_{\mathbb{R}} \frac{f(\varepsilon\sqrt{t}y)}{\sqrt{2\pi}}\exp\left\{\frac{t^{(\gamma+1)/2}}{(\gamma+1)\varepsilon^{1-\gamma}}|y|^{\gamma+1} - \frac{y^2}{2}\right\}dy$$

$$\times\mathbb{E}\left[\exp\left\{-\frac{\gamma t^{(\gamma+1)/2}}{2\varepsilon^{1-\gamma}}\int_0^1 |ys+b_s|^{\gamma-1}ds - \frac{t^{\gamma+1}}{2\varepsilon^{2-2\gamma}}\int_0^1 |ys+b_s|^{2\gamma}ds\right\}\right].$$

By the change of variable $x = \varepsilon\sqrt{t}y$, the above formula becomes

$$\mathbb{E}[f(X_t^\varepsilon)] = \int_{\mathbb{R}} \frac{f(x)}{\varepsilon\sqrt{2\pi t}}\exp\left\{\frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t}\right\}$$

$$\times\mathbb{E}\left[\exp\left\{-\frac{\gamma t}{2}\int_0^1 |xs+\varepsilon\sqrt{t}b_s|^{\gamma-1}ds - \frac{t}{2\varepsilon^2}\int_0^1 |xs+\varepsilon\sqrt{t}b_s|^{2\gamma}ds\right\}\right]dx$$

and we obtain the expression of the density (5). **QED**

Another useful expression of a density is contained in the following:

Corollary 1 *For $t>0$, $\varepsilon>0$ and $x\in\mathbb{R}$,*

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon\sqrt{2\pi t}}\exp\left\{\frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t}\right\}\mathbb{E}_{\frac{x}{\varepsilon(s(\varepsilon))^{1/2}}}\left[\exp\left\{-\int_0^{t/s(\varepsilon)}\frac{V(B_s)}{2}ds\right\}\middle| B_{\frac{t}{s(\varepsilon)}} = 0\right] \quad (8)$$

where we denoted $s(x) = |x|^{(2(1-\gamma))/(1+\gamma)}$ and the potential V is given by:

$$V(x) := \frac{\gamma}{|x|^{1-\gamma}} + |x|^{2\gamma}. \quad (9)$$

Proof: By conditioning with respect to $\{B_t = x\}$ in (7) we obtain

$$\mathbb{E}[f(X_t^\varepsilon)] = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}}f(\varepsilon x)\exp\left\{\frac{1}{(\gamma+1)\varepsilon^{1-\gamma}}|x|^{\gamma+1} - \frac{x^2}{2t}\right\}dx$$

$$\times\mathbb{E}_0\left[\exp-\left\{\frac{\gamma}{2\varepsilon^{1-\gamma}}\int_0^t |B_s|^{\gamma-1}ds - \frac{1}{2\varepsilon^{2-2\gamma}}\int_0^t |B_s|^{2\gamma}ds\right\}\middle| B_t = x\right].$$

The functional of the Brownian motion which appears in the integral on the right hand side of the previous equality is time homogeneous. By the time-reversal property, we can replace the

Brownian motion B_s starting from 0 and going to x in time t by a Brownian motion starting from x and going to 0 in time t. Therefore we get:

$$\mathbb{E}_0 \left[\exp - \left\{ \frac{\gamma}{2\varepsilon^{1-\gamma}} \int_0^t |B_s|^{\gamma-1} ds - \frac{1}{2\varepsilon^{2-2\gamma}} \int_0^t |B_s|^{2\gamma} ds \right\} \Big| B_t = x \right]$$

$$= \mathbb{E}_x \left[\exp - \left\{ \frac{\gamma}{2\varepsilon^{1-\gamma}} \int_0^t |B_s|^{\gamma-1} ds - \frac{1}{2\varepsilon^{2-2\gamma}} \int_0^t |B_s|^{2\gamma} ds \right\} \Big| B_t = 0 \right]$$

By scaling we get (8). **QED**

The following result contains an expansion of the density of X_t^ε in terms of the eigenvalues and the eigenfunctions of a Schrödinger operator. This type of expression was already considered in [K], p. 194 for continuous potentials.

Proposition 5 *For $t > 0$, $\varepsilon > 0$ and $x \in \mathbb{R}$:*

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}} \exp \left\{ \frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} \right\} \sum_{j=1}^{\infty} e^{-\frac{\lambda_j t}{s(\varepsilon)}} \psi_j(0) \psi_j \left(\frac{|x|}{\varepsilon s(\varepsilon)^{1/2}} \right), \tag{10}$$

where λ_j and ψ_j are the eigenvalues and the eigenfunctions of the operator on $L^2(\mathbb{R})$:

$$-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}V(x).$$

Moreover the series is uniform convergent for fixed t and x belonging to a compact set of $\mathbb{R}$.

Proof: Let us consider the following one-parameter semi-group:

$$(T_t f)(x) := \mathbb{E}_x \left[f(B_t) \exp -\frac{1}{2} \int_0^t V(B_s) ds \right],$$

and we shall denote by $a_t(x,y)$ the density of the semi-group with respect to the Lebesgue measure:

$$a_t(x,y) = \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} \mathbb{E}_x \left[\exp -\frac{1}{2} \int_0^t V(B_s) ds \Big| B_t = y \right].$$

Therefore, by (8) we can write

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}} \exp \left\{ \frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} \right\} a_{\frac{t}{s(\varepsilon)}} \left(0, \frac{x}{\varepsilon s(\varepsilon)^{1/2}} \right).$$

Let us note that, by the definition, the semi-group T_t preserve the positivity. Moreover, the generator of $T_t = e^{-Ht}$ is $-H$, with H a positive self-adjoint operator. Indeed, this second property is true for self-adjoint contraction semi-groups (see, [D2] Theorem 4.6, p. 99) and we can prove that

$$\|T_t\|_{L^2(\mathbb{R}),L^2(\mathbb{R})} \leq \sup_{x\in\mathbb{R}} \mathbb{E}_x \left[\exp -\frac{1}{2} \int_0^t V(B_s) ds \right] \leq 1,$$

since $V(\cdot) \geq 0$ (see, for instance, [Ca], p. 271).
It can be shown (see [D-S] Lemma 2.1 p. 339) that the density of a semi-group satisfying the previous properties and which is a trace class operator, can be developed as

$$a_t(x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \psi_j(x) \psi_j(y).$$

Here the λ_j and ψ_j are the eigenvalues and the eigenfunctions of the discrete spectrum of the equation

$$-\frac{1}{2}\psi''(x) - \frac{1}{2}V(x)\psi(x) = \lambda\psi(x).$$

Moreover the convergence of the series is uniform over all compact sets of $\mathbb{R}\times\mathbb{R}$.
To obtain the result (10) we shall prove that T_t is a trace class operator. Clearly,

$$a_t(x,x) \leq \frac{1}{\sqrt{2\pi t}}\mathbb{E}\left[\exp -\frac{1}{2}\int_0^t |x+B_s^{0,t}|^{2\gamma}ds\right] =: \tilde{a}_t(x,x), \tag{11}$$

where $B_s^{0,t}$ is the Brownian bridge from 0 to 0 over $[0,t]$ (thus the standard Brownian bridge is $b_s = B_s^{0,1}$) and $\tilde{a}_t(x,y)$ is the density of the semi-group generated by the Schrödinger operator

$$-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\tilde{V}(x), \text{ with } \tilde{V}(x) = |x|^{2\gamma}.$$

Since, $\tilde{V} \in L^2_{\mathrm{loc}}(\mathbb{R})$ and $\lim_{x\to\infty}\tilde{V}(x)/|x|^\gamma = +\infty$ we can deduce that this operator is a class trace operator (see also [D1] Theorem 3.2, p. 488). By Mercer's theorem (see, for instance, [R-S], p. 65), we get

$$\int_{\mathbb{R}} \tilde{a}_t(x,x)dx < \infty,$$

and then, by (11),

$$\int_{\mathbb{R}} a_t(x,x)dx < \infty.$$

Again by Mercer's theorem, we deduce that T_t is a trace class operator. **QED.**

Remark 2 In the particular case $\gamma = 1/2$ we can find an equivalent of $\psi_j(x)$:

$$\psi_j(x) \sim \exp\{-\frac{2}{3}x^{3/2} + 2\sqrt{x}\lambda_j\}, \text{ as } x\uparrow\infty.$$

This enables to think that $\varepsilon^{2/3}\ln p_t^\varepsilon(x)$ tends to $\lambda_1(2\sqrt{|x|}-t)$, if (t,x) lies between the extremal solutions $\rho_{1,2}(t) = \pm t^2/4$.

The second part of this remark can be proved in the following simple case $x = \varepsilon s(\varepsilon)^{1/2}$:

Corollary 2 *For $t>0$,*

$$\lim_{\varepsilon\to 0} s(\varepsilon)\ln p_t^\varepsilon(\varepsilon s(\varepsilon)^{1/2}) = -\lambda_1 t. \tag{12}$$

Moreover, the convergence is uniform on any compact subset of $\mathbb{R}_+^*$.

Proof: By (10) we get

$$p_t^\varepsilon(\varepsilon s(\varepsilon)^{1/2}) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}}\exp\left\{\frac{1}{(\gamma+1)}\right\}\sum_{j=1}^\infty e^{-\frac{\lambda_j t}{s(\varepsilon)}}\psi_j(0)\psi_j(1)$$

Since V is bounded from below, there exists a constant $K>0$ such that

$$\text{for all } j\geq 1, \quad \|\psi_j\|_\infty \leq K\|\psi_j\|_2$$

(see [D1] Lemma 3.1, p. 488). Therefore, by classical convergence theorems,

$$p_t^\varepsilon(\varepsilon s(\varepsilon)^{1/2}) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}}\exp\left\{\frac{1}{(\gamma+1)} - \frac{\lambda_1 t}{s(\varepsilon)}\right\}(\psi_1(0)\psi_1(1) + o(s(\varepsilon)))$$

and we obtain the announced result. It is not difficult to modify this proof to obtain the uniform convergence. **QED**

3. CONVERGENCE OF $\varepsilon^2 \ln p_t^\varepsilon(x)$

The purpose of this section is to study the behaviour of $\varepsilon^2 \ln p_t^\varepsilon(x)$. The result will be sharp if (t,x) does not lie between the two extremal solutions of (3).

Theorem 1 *If $|x| > \{t(1-\gamma)\}^{1/1-\gamma}$ then there exists a positive function k_t such that*

$$\lim_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x) = -k_t(|x|). \tag{13}$$

Remark 3 We also prove that if $|x| \le \{t(1-\gamma)\}^{1/1-\gamma}$ then

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x) \le 0, \tag{14}$$

but this result will be improved in §5.

To prove the theorem we need to apply the large deviations theory and in particular the Varadhan principle. Let us first introduce the spaces:

$$H^1 := \left\{ \int_0^t f(s)ds : f \in L^2([0,1]) \right\},\ H_0^1 := \left\{ f \in H^1 : f(1) = 0 \right\},$$

endowed with the norm

$$\|f\|_{H^1} := \left(\int_0^1 |f'(s)|^2 ds \right)^{1/2}.$$

Proposition 6 *Let us consider, for $\phi \in H^1$, $t > 0$, $\quad 0 < \gamma < 1$:*

$$A(\phi) = t \int_0^1 |xu - \sqrt{t}\phi(u)|^{2\gamma} du + \int_0^1 \phi'^2(u) du. \tag{15}$$

Then

$$\inf_{\phi\in H_0^1} A(\phi) = \begin{cases} \frac{2|x|^{1+\gamma}}{\gamma+1} - \frac{x^2}{t}, & \text{if } |x| \le \{t(1-\gamma)\}^{1/(1-\gamma)} \\ \frac{2|x|^{1+\gamma}}{\gamma+1} - \frac{x^2}{t} + 2k_t(|x|), & \text{else.} \end{cases} \tag{16}$$

Proof: First, we can assume that $x \ge 0$. Indeed, if $x \le 0$ it suffices to replace in (5) b_u by $-b_u$ which are identical in law, to obtain the result.
i) Let us denote

$$a = \sup\left\{ 0 \le u \le 1 : \phi(u) = \frac{xu}{\sqrt{t}} \right\}. \tag{17}$$

It is obvious that on [0,1], the straight line $l(u) := xu/\sqrt{t}$ minimise the functional

$$\phi \mapsto \int_0^1 |xs - \sqrt{t}\phi(s)|^{2\gamma} ds.$$

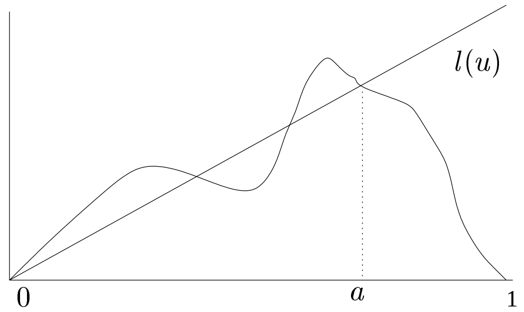

Figure 3: Description of a

Moreover
$$\int_0^a \phi'^2(u)du \geq \frac{ax^2}{2} = \int_0^a (xu/t)'^2 du, \ \forall \phi \in H^1.$$
Indeed
$$\frac{ax}{\sqrt{t}} = \phi(a) = |\int_0^a \phi'(u)du| \leq \sqrt{a}\left(\int_0^a \phi'^2(u)du\right)^{1/2}.$$
ii) Let $\phi_0 \in H_0^1$ be such that $A(\phi_0) = \inf A(\phi)$. By **i)** we see that on [0,a], $\phi_0 = l$. Let us notice also that
$$\phi_0(u) \leq \frac{xu}{\sqrt{t}}. \tag{18}$$
For any $h \in H^1$ such that $h(0) = h(1) = 0$
$$\frac{d}{d\lambda}A(\phi_0 + \lambda h) = 0.$$
Integrating by parts the previous equality, we get by (15)
$$\int_0^1 \gamma t^{\frac{3}{2}} |xu - \sqrt{t}\phi_0(u)|^{2\gamma-1}\mathrm{sgn}(xu - \sqrt{t}\phi_0(u))h(u)du + \int_0^1 \phi_0''(u)h(u)du = 0. \tag{19}$$
Let us denote, for $u \in [0,1]$,
$$y(u) := xu - \sqrt{t}\phi_0(u) \geq 0,$$
by (18). We shall prove that
$$\begin{cases} y''(u) = \gamma t^2 y^{2\gamma-1}(u), & \text{on } [a,1] \\ y(a) = 0, y(1) = x. \end{cases} \tag{20}$$
Since h is arbitrary, by (19):
$$\gamma t^{\frac{3}{2}} |xu - \sqrt{t}\phi_0(u)|^{2\gamma-1}\mathrm{sgn}(xu - \sqrt{t}\phi_0(u)) + \phi_0''(u) = 0, \forall u \in [0,1],$$
or, by (18):
$$\phi_0''(u) = -\gamma t^2 (xu - \sqrt{t}\phi_0(u))^{2\gamma-1}.$$
Since $y''(u) = -\sqrt{t}\phi_0''(u)$, we get (20).

iii) Assume $a > 0$. Then
$$y'(a) = 0. \tag{21}$$
Indeed, by (19), since $\phi_0''(u) = 0, \forall u \in [0,a]$, we can write
$$0 = \int_0^1 \gamma t^{\frac{3}{2}} |xu - \sqrt{t}\phi_0(u)|^{2\gamma-1}\mathrm{sgn}(xu - \sqrt{t}\phi_0(u))h(u)du + \int_a^1 \phi_0''(u)h(u)du$$

$$= \int_0^1 \gamma t^{\frac{3}{2}} |xu - \sqrt{t}\phi_0(u)|^{2\gamma-1} \mathrm{sgn}(xu - \sqrt{t}\phi_0(u)) h(u) du - \phi_0'(a) h(a) - \int_a^1 \phi_0'(u) h'(u) du$$

$$= \frac{1}{2} \frac{d}{d\lambda} A(\phi_0 + \lambda h) + \int_0^a \phi_0'(u) h'(u) du - \phi_0'(a) h(a).$$

Hence, by **i)**

$$\int_0^a \frac{x}{\sqrt{t}} h'(u) du = \phi_0'(a) h(a),$$

which is equivalent to $\phi_0'(a) = x/\sqrt{a}$ and (21) is proved.

iv) We shall prove that, for $a > 0$ the solution of the problem (20), (21) satisfies:

$$u = a + \frac{y(u)^{1-\gamma}}{t(1-\gamma)}. \tag{22}$$

Let us determine $y(\cdot)$ on [a,1]:

$$\frac{d(y')^2}{du} = 2y'y'' = 2y'(\gamma t^2 y^{2\gamma-1}) = 2\gamma t^2 y^{2\gamma-1} y'\,.$$

Therefore

$$\begin{aligned} (y'(u))^2 &= (y'(a))^2 + 2\gamma t^2 \int_a^u y(x)^{2\gamma-1} y'(x) dx \\ &= (y'(a))^2 + 2\gamma t^2 \int_{y(a)}^{y(u)} x^{2\gamma-1} dx \\ &= (y'(a))^2 + t^2 y(u)^{2\gamma} - t^2 y(a)^{2\gamma}. \end{aligned} \tag{23}$$

Since $y(a) = y'(a) = 0$ and $y'(u) \geq 0$ we obtain:

$$\frac{dy}{du} = t y^{\gamma} \iff \frac{du}{dy} = \frac{1}{t y^{\gamma}},$$

or,

$$u = a + \int_{y(a)}^{y(u)} \frac{dx}{t x^{\gamma}} = a + \frac{y(u)^{1-\gamma}}{t(1-\gamma)}.$$

Finally, since $y(1) = x$ we get

$$a = 1 - \frac{x^{1-\gamma}}{t(1-\gamma)}, \tag{24}$$

and the condition $a > 0$ can be written as

$$x < \{t(1-\gamma)\}^{1/(1-\gamma)},$$

namely (t, x) lies between the two extremals.
v) We need to compute the minimum of A

$$\inf A(\phi) = A(\phi_0) = A(y(\cdot)/\sqrt{t} - l(\cdot)) =$$

$$t \int_0^1 |xu - \sqrt{t}\phi_0(u)|^{2\gamma} du + \int_0^1 \phi_0'^2(u) du = \frac{x^2 a}{t} + t \int_a^1 |xu - \sqrt{t}\phi_0(u)|^{2\gamma} du + \int_a^1 \phi_0'^2(u) du.$$

Since $\phi_0(u) = xu/\sqrt{t}$ on [0,a],

$$\begin{aligned} A(\phi_0) &= \frac{x^2 a}{t} + t\int_a^1 y(u)^{2\gamma}du + \int_a^1 \frac{(x-y'(u))^2}{t}du \\ &= \frac{x^2 a}{t} + t\int_a^1 y(u)^{2\gamma}du + \frac{x^2(1-a)}{t} - \frac{2x}{t}\int_a^1 y'(u)du + \frac{1}{t}\int_a^1 y'^2(u)du. \end{aligned}$$

By (22) we obtain:

$$A(\phi_0) = \frac{x^2}{t} + 2t\int_a^1 \{t(1-\gamma)(u-a)\}^{\frac{2\gamma}{1-\gamma}}du - 2x\int_a^1 \{t(1-\gamma)(u-a)\}^{\frac{\gamma}{1-\gamma}}du,$$

which can be written, by change of variable $v = t(1-\gamma)(u-a)$ and by (24), as

$$A(\phi_0) = \frac{x^2}{t} + \frac{2}{1-\gamma}\int_0^{x^{1-\gamma}} v^{\frac{2\gamma}{1-\gamma}}dv - \frac{2x}{t(1-\gamma)}\int_0^{x^{1-\gamma}} v^{\frac{\gamma}{1-\gamma}}dv.$$

Then we get the first part of (16) by straightforward calculation.

vi) Assume now $a = 0$ which means, that:

$$x \geq (t(1-\gamma))^{1/(1-\gamma)}.$$

As in **iv)**, the solution of the problem (20) satisfies

$$y'(u)^2 = t^2 y(u)^{2\gamma} + y'(0)^2.$$

However in this case (a=0) we have not the explicit value of $y'(0)$, as in (21).

vii) We need to compute the minimum of A

$$\inf A(\phi) = A(\phi_0)$$

$$\begin{aligned} &= t\int_0^1 y(u)^{2\gamma}du + \int_0^1 \frac{(x-y'(u))^2}{t}du \\ &= \frac{2}{t}\int_0^1 y'(u)^2du - \frac{y'(0)^2}{t} + \frac{x^2}{t} - \frac{2x}{t}\int_0^1 y'(u)du \\ &= \frac{2}{t}\int_0^1 y'(u)^2du - \frac{y'(0)^2}{t} - \frac{x^2}{t}. \end{aligned}$$

Therefore

$$A(\phi_0) = \frac{2}{t}\int_0^x \sqrt{y'(0)^2 + t^2 y^{2\gamma}}dy - \frac{y'(0)^2}{t} - \frac{x^2}{t}, \tag{25}$$

since

$$\frac{du}{dy} = \frac{1}{\sqrt{y'(0)^2 + t^2y^{2\gamma}}}.$$

Recall that $y(1) = x$, hence:

$$1 = \int_0^x \frac{du}{dy}dy = \int_0^x \frac{dy}{\sqrt{y'(0)^2 + t^2y^{2\gamma}}}.$$

If $y'(0) = 0$, then

$$1 = \int_0^x \frac{dy}{ty^\gamma} = \frac{x^{1-\gamma}}{t(1-\gamma)} \Leftrightarrow x = \{t(1-\gamma)\}^{1/(1-\gamma)}.$$

But $x > \{t(1-\gamma)\}^{1/(1-\gamma)}$ and, since $y \geq 0$, we have necessary $y'(0) > 0$. By integration by parts:

$$\begin{aligned} I &:= \int_0^x \sqrt{y'(0)^2 + t^2 y^{2\gamma}} dy = \left[y\sqrt{y'(0)^2 + t^2 y^{2\gamma}} \right]_{y=0}^{y=x} - \gamma \int_0^x \frac{t^2 y^{2\gamma}}{\sqrt{y'(0)^2 + t^2 y^{2\gamma}}} dy \\ &= x\sqrt{y'(0)^2 + t^2 x^{2\gamma}} - \gamma \int_0^x \sqrt{y'(0)^2 + t^2 y^{2\gamma}} dy + \gamma y'(0)^2 \int_0^x \frac{dy}{\sqrt{y'(0)^2 + t^2 y^{2\gamma}}} \\ &= x\sqrt{y'(0)^2 + t^2 x^{2\gamma}} - \gamma I + \gamma y'(0)^2. \end{aligned}$$

Hence

$$I = \frac{x\sqrt{y'(0)^2 + t^2 x^{2\gamma}}}{1+\gamma} + \frac{\gamma}{1+\gamma} y'(0)^2.$$

Replacing I in (25) we obtain

$$\begin{aligned} A(\phi_0) &= \frac{2x\sqrt{y'(0)^2 + t^2 x^{2\gamma}}}{(1+\gamma)t} + \frac{(\gamma-1)y'(0)^2}{(1+\gamma)t} - \frac{x^2}{t} \\ &= \frac{2x^{\gamma+1}}{1+\gamma} - \frac{x^2}{t} + 2k_t(x), \end{aligned}$$

where

$$t(1+\gamma)k_t(x) := x\sqrt{y'(0)^2 + t^2 x^{2\gamma}} - tx^{1+\gamma} + \frac{\gamma-1}{2} y'(0)^2.$$

Let us prove that $k_t(x) > 0$. By the classical inequality $\sqrt{a+b} - \sqrt{b} \geq \sqrt{a}/2$, we have

$$\sqrt{y'(0)^2 + t^2 x^{2\gamma}} - tx^\gamma \geq \frac{y'(0)}{2}.$$

Recall that, $x > \{t(1-\gamma)\}^{1/(1-\gamma)}$ is equivalent to $y'(0) > 0$. Therefore

$$t(1+\gamma)k_t(x) \geq \frac{xy'(0)}{2} + \frac{(\gamma-1)y'(0)^2}{2} > 0.$$

This ends the proof of the second part of (16). **QED**

Proof of Theorem 1: Clearly, by (5) we can write

$$p_t^\varepsilon(x) = \frac{1}{\varepsilon\sqrt{2\pi t}} \exp\left\{ \frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t} \right\} \mathbb{E}\left[\exp\left(-F(\varepsilon b) - \frac{G(\varepsilon b)}{\varepsilon^2} \right) \right],$$

where

$$F(\phi) = \frac{\gamma t}{2} \int_0^1 |xu - \sqrt{t}\phi(u)|^{\gamma-1} du$$

and

$$G(\phi) = \frac{t}{2} \int_0^1 |xu - \sqrt{t}\phi(u)|^{2\gamma} du.$$

i) (an upper bound for $\limsup_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x)$)

We have

$$p_t^\varepsilon(x) \leq \frac{1}{\varepsilon\sqrt{2\pi t}} \exp\left\{ \frac{|x|^{\gamma+1}}{(\gamma+1)\varepsilon^2} - \frac{x^2}{2\varepsilon^2 t} \right\} \mathbb{E}\left[\exp - \frac{G(\varepsilon b)}{\varepsilon^2} \right] =: r_t^\varepsilon(x).$$

The functional G is a continuous lower bounded functional of the Brownian motion. Therefore, to study $r_t^\varepsilon(x)$ we use the Varadhan principle (see for instance [De-S], p. 43). Hence, applying the logarithm we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \ln p_t^\varepsilon(x) \leq \frac{|x|^{\gamma+1}}{\gamma+1} - \frac{x^2}{2t} - \frac{1}{2} \inf_{\phi\in H_0^1} A(\phi).$$

Using (16) we deduce (14) and

$$\limsup_{\varepsilon\to 0}\varepsilon^2\ln p_t^\varepsilon(x)\le -k_t(|x|).$$

ii) (a lower bound for $\liminf_{\varepsilon\to 0}\varepsilon^2\ln p_t^\varepsilon(x)$)
We assume that $x>\{t(1-\gamma)\}^{1/1-\gamma}$ and let us denote $\kappa:=\frac{1}{2}(x-\sqrt{t}\phi_0'(0))>0$. Here ϕ_0 is the function which minimise the functional A (see the proof of Proposition 6). Since $\phi_0''\le 0$, ϕ_0 belongs to the following open set

$$\mathcal{U}:=\{\phi\in\mathcal{C}: xu-\sqrt{t}\phi(u)>\kappa\, u,\ \forall u\in[0,1]\}.$$

Moreover, there exists $\eta>0$ such that

$$\max_{\phi\in\mathcal{U}}F(\phi)\le\eta.$$

Take $\delta>0$ and let $\mathcal{V}$ be a neighbourhood of ϕ_0 such that

$$\max_{\phi\in\mathcal{V}}G(\phi)\le G(\phi_0)+\delta.$$

Let us denote $\mathcal{W}:=\mathcal{U}\cap\mathcal{V}$. Then we can write

$$\begin{aligned}\liminf_{\varepsilon\to 0}\varepsilon^2\ln\mathbb{E}\left[\exp-\left(F(\varepsilon b)+\frac{G(\varepsilon b)}{\varepsilon^2}\right)\right] &\ge \liminf_{\varepsilon\to 0}\varepsilon^2\ln\mathbb{E}\left[\exp-\left(F(\varepsilon b)+\frac{G(\varepsilon b)}{\varepsilon^2}\right)\mathbb{I}_{\{\varepsilon b\in\mathcal{W}\}}\right]\\ &\ge \liminf_{\varepsilon\to 0}\varepsilon^2\ln\mathbb{P}(\varepsilon b\in\mathcal{W})-\liminf_{\varepsilon\to 0}\varepsilon^2\eta-\max_{\phi\in\mathcal{W}}G(\phi).\end{aligned}$$

By Schilder's theorem (see for instance [D-S], p 18), we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0}\varepsilon^2\ln\mathbb{E}\left[\exp-\left(F(\varepsilon b)+\frac{G(\varepsilon b)}{\varepsilon^2}\right)\right] &\ge -\inf_{\phi\in\mathcal{W}\cap H_0^1}\frac{1}{2}\int_0^1|\phi'(u)|^2du-G(\phi_0)-\delta\\ &\ge -\frac{1}{2}A(\phi_0)-\delta.\end{aligned}$$

Letting $\delta\to 0$ we get

$$\liminf_{\varepsilon\to 0}\varepsilon^2\ln p_t^\varepsilon(x)\ge\frac{|x|^{\gamma+1}}{\gamma+1}-\frac{x^2}{2t}-\frac{1}{2}\inf_{\phi\in H_0^1}A(\phi).$$

QED

4. STUDY OF THE PARTICULAR CASE $\gamma=1/2$

In this section we shall obtain an upper bound for $\limsup_{\varepsilon\to 0}\varepsilon^{2/3}\ln p_t^\varepsilon(x)$, for (t,x) lying in the domain contained between the extremal solutions.

Theorem 2 *For $|x|\le t^2/4$,*

$$\limsup_{\varepsilon\to 0}\varepsilon^{2/3}\ln p_t^\varepsilon(x)\le a_1'(t/2-\sqrt{|x|}),\tag{26}$$

where a_1' is the greater negative zero of the derivative of the Airy function Ai.

Remark 4 It is known that the value of a'_1 is: $a'_1 = -1.01879297...$ (see, for instance, [A-S], p. 172). Let us note that in this result we get only an inequality, since we bound by 1 the singular factor

$$\exp\left\{-\frac{t}{4}\int_0^1 |xs+\varepsilon\sqrt{t}b_s|^{-1/2}ds\right\}$$

in the expression of p_t^ε in (5). In §5 we shall improve this result, by computing the limit of $\varepsilon^{2/3}\ln p_t^\varepsilon(x)$ and also the precise constant instead a'_1.

To prove this theorem we shall use the following result concerning a functional of the standard Brownian bridge $\{b_u, u\in[0,1]\}$, which can be interesting in itself:

Proposition 7 *For* $0\le a<1$,

$$\lim_{\varepsilon\to 0}\varepsilon^{2/3}\ln\mathbb{E}\left[\exp-\frac{1}{\varepsilon}\int_0^a |b_u|du\right] = \frac{a'_1 a}{2^{1/3}}\,. \tag{27}$$

Proof of Proposition 7: We denote by $B_t^{x,\delta}$ a Brownian bridge from 0 to x over $[0,\delta]$. By the scaling property of the Brownian motion, we get

$$\mathbb{E}\left[\exp-\frac{1}{\varepsilon}\int_0^a |B_u^{x,a}|du\right] = \mathbb{E}\left[\exp-\frac{1}{\varepsilon}\int_0^{a/\varepsilon^{2/3}} |B_u^{x/\varepsilon^{1/3},a/\varepsilon^{2/3}}|du\right]. \tag{28}$$

Conditioning $\{b_u : 0\le u\le a\}$ with respect to the centred Gaussian variable b_a, which have the variance $\sigma = a(1-a)$, we obtain

$$\begin{aligned}\mathbb{E}\left[\exp-\frac{1}{\varepsilon}\int_0^a |b_u|du\right] &= \int_0^\infty\sqrt{\frac{2}{\sigma\pi}}\mathbb{E}\left[\exp-\frac{1}{\varepsilon}\int_0^a |B_u^{x,a}|du\right]\exp-\frac{x^2}{2\sigma}dx\\ &= \int_0^\infty\sqrt{\frac{2}{\sigma\pi}}\mathbb{E}\left[\exp-\frac{1}{\varepsilon}\int_0^{a/\varepsilon^{2/3}} |B_u^{x/\varepsilon^{1/3},a/\varepsilon^{2/3}}|du\right]\exp-\frac{x^2}{2\sigma}dx.\end{aligned}$$

We are going to calculate:

$$\mathbb{E}\left[\exp-\int_0^t |B_u^{x,t}|du\right]$$

using similar arguments as in [Sh].
First, let us recall Kac's formula (see for instance [I-MK], p. 54): for any bounded function with compact support f, and any potential $V\ge 0$,

$$\mathbb{E}\left[\int_0^\infty \exp-\left\{\alpha t+\int_0^t V(B_s)ds\right\}f(B_t)dt\right] = u(0), \tag{29}$$

where u is the unique bounded solution of the equation

$$-\frac{1}{2}u''(x)+(\alpha+V(x))u(x) = f(x). \tag{30}$$

Take $V(x)=|x|$ and let Φ, Ψ be two solutions of the homogeneous equation

$$-\frac{1}{2}u''(x)+(\alpha+V(x))u(x) = 0,$$

such that Φ is bounded at $+\infty$, Ψ is bounded at $-\infty$, and such that

$$\Phi\Psi'-\Phi'\Psi = 2. \tag{31}$$

Then the solution of the equation (30) is obtained applying the Green operator to f:

$$u(x) = \Phi(x) \int_{-\infty}^{x} \Psi(y) f(y) dy + \Psi(x) \int_{x}^{\infty} \Phi(y) f(y) dy, \tag{32}$$

and, by (29)

$$\mathbb{E}\left[\int_0^\infty \exp - \left\{\alpha t + \int_0^t |B_s| ds\right\} f(B_t) dt\right] = \Phi(0) \int_{-\infty}^{0} \Psi(y) f(y) dy + \Psi(0) \int_{0}^{\infty} \Phi(y) f(y) dy. \tag{33}$$

It is known (see, for instance, [A-S], p. 162) that the Airy's functions Ai and Bi satisfy $-u''(x) + x\,u(x) = 0$, with $Ai(\cdot)$ bounded at $+\infty$. Therefore we get:

$$\Phi(x) = \begin{cases} d_0 Ai(2^{1/3}(x+\alpha)), & \text{if } x \geq 0 \\ d_1 Ai(2^{1/3}(-x+\alpha)) + d_2 Bi(2^{1/3}(-x+\alpha)), & \text{if } x < 0 \end{cases} = \Psi(-x), \tag{34}$$

where the last equality is obtained by symmetry. Then the constants d_0, d_1, d_2 can be determined using the fact that $\Phi(0+) = \Phi(0-)$, $\Phi'(0+) = \Phi'(0-)$ and by (31). We obtain, for instance,

$$d_0^2 = -\frac{2^{-1/3}}{Ai(2^{1/3}\alpha) Ai'(2^{1/3}\alpha)}. \tag{35}$$

We need to condition $\{B_s : 0 \leq s \leq t\}$ with respect to $B_t = x$. Therefore we take

$$f(y) = \frac{\sqrt{2\pi}}{2\delta} \mathbb{1}_{\{|y-x|<\delta\}}(y)$$

and we let $\delta \downarrow 0$ in (33). On the right side of (33) we get

$$\begin{cases} \sqrt{2\pi}\Psi(0)\Phi(x), & \text{if } x \geq 0 \\ \sqrt{2\pi}\Phi(0)\Psi(x), & \text{if } x < 0 \end{cases} = \sqrt{2\pi}\Phi(0)\Phi(|x|) = \sqrt{2\pi} d_0^2 Ai(2^{1/3}(|x|+\alpha)) Ai(2^{1/3}\alpha)$$

$$= -\frac{\sqrt{2\pi} 2^{-1/3} Ai(2^{1/3}(|\xi|+\alpha))}{Ai'(2^{1/3}\alpha)},$$

by (34) and (35). The limit on the left side of (33) can be written as:

$$\lim_{\delta \downarrow 0} \mathbb{E}\left[\int_0^\infty \exp - \left\{\alpha t + \int_0^t |B_s| ds\right\} \frac{\mathbb{1}_{\{|B_t - x|<\delta\}}}{\mathbb{P}(|B_t - x| < \delta)} \frac{\mathbb{P}(|B_t - \xi| < \delta)}{2\delta/\sqrt{2\pi t}} \frac{dt}{\sqrt{t}}\right] \tag{36}$$

We note that $\mathbb{P}(|B_t - x| < \delta) \leq \frac{2\delta}{\sqrt{2\pi t}}$ and $\lim_{\delta \downarrow 0} \dfrac{\mathbb{P}(|B_t - x| < \delta)}{2\delta/\sqrt{2\pi t}} = \exp -\dfrac{x^2}{2t}$. Then, by the Lebesgue dominated theorem we obtain for the limit (36)

$$\int_0^\infty e^{-\alpha t} \mathbb{E}\left[\exp - \int_0^t |B_s| ds \,\middle|\, B_t = x\right] \exp\left(-\frac{x^2}{2t}\right) \frac{dt}{\sqrt{t}} = \frac{\sqrt{2\pi} 2^{-1/3} Ai(2^{1/3}(|x|+\alpha))}{Ai'(2^{1/3}\alpha)}.$$

Inverting the Laplace transform (see, for instance, [Ri], p. 241), we deduce

$$\begin{aligned} \mathbb{E}\left[\exp - \int_0^t |B_s| ds \,\middle|\, B_t = x\right] \frac{\exp -\frac{x^2}{2t}}{\sqrt{t}} &= -\sqrt{\pi} 2^{1/6} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{us} Ai(2^{1/3}(|x|+u))}{Ai'(2^{1/3}u)} du \\ &= -\sqrt{\pi} 2^{1/6} \sum_{n=1}^{\infty} \frac{Ai(2^{1/3}|\xi| + a_n')}{Ai''(a_n')} \exp \frac{a_n' t}{2^{1/3}}, \end{aligned}$$

for x such that $Ai(2^{1/3}|x|+a'_n)\neq 0$, $\forall n$. Here a'_n is the n-th zero of Ai'. The last equality was obtained by calculating the residues of $1/Ai'(u)$.
Finally, since $-Ai''(x)+x\,Ai(x)=0$,

$$\mathbb{E}\left[\exp-\int_0^t|B_s|ds\bigg|\,B_t=x\right]\frac{\exp-\frac{x^2}{2t}}{\sqrt{t}}=-\sqrt{\pi}2^{1/6}\sum_{n=1}^{\infty}\frac{Ai(2^{1/3}|x|+a'_n)}{a'_nAi(a'_n)}\exp\frac{a'_nt}{2^{1/3}}$$

To obtain $\mathbb{E}\left[\exp-\frac{1}{\varepsilon}\int_0^a|b_u|du\right]$ we need to integrate the previous equality:

$$\begin{aligned}
&\int_0^\infty\frac{1}{\sigma\sqrt{2\pi}}\mathbb{E}\left[\exp-\int_0^{a/\varepsilon^{2/3}}|B_u^{x/\varepsilon^{1/3},a/\varepsilon^{2/3}}|du\right]\exp-\frac{x^2}{2\sigma^2}dx\\
=&\ -\frac{2^{2/3}}{\sigma}\int_0^\infty\sum_{n=1}^{\infty}\frac{Ai(2^{1/3}|x/\varepsilon^{1/3}|+a'_n)}{a'_nAi(a'_n)}\exp\frac{a'_na}{2^{1/3}\varepsilon^{2/3}}\exp(\frac{1}{a}-\frac{1}{\sigma^2})\frac{x^2}{2}dx\\
=&\ \frac{2^{2/3}}{\sigma}\sum_{n=1}^{\infty}c_n\exp\frac{a'_na}{2^{1/3}\varepsilon^{2/3}},
\end{aligned}$$

where

$$c_n=\frac{1}{a'_nAi(a'_n)}\int_0^\infty Ai(2^{1/3}|\frac{x}{\varepsilon^{1/3}}|+a'_n)\exp\left\{(\frac{1}{a}-\frac{1}{\sigma^2})\frac{x^2}{2}\right\}dx.$$

Indeed we have $\sigma^2=a(1-a)<a$ and $|Ai(u)|\leq Ai(a'_1)$ (see, for instance, [A-S], p. 162); moreover $|Ai(a'_n)|\sim b_1n^{-1/6}$ and $-a'_n\sim b_2n^{2/3}$, as $n\uparrow\infty$, where b_1 and b_2 are some constants. By the Raabe-Duhamel criterion we obtain the convergence of the series.
Furthermore, by classical arguments of dominated and monotone convergence we can write

$$\mathbb{E}\left[\exp-\frac{1}{\varepsilon}\int_0^a|b_u|du\right]=c_1\frac{2^{2/3}}{a(1-a)}\exp(\frac{a'_1a}{2^{1/3}\varepsilon^2/3})\left(1+o(\frac{1}{\varepsilon^{2/3}})\right),$$

and we deduce (27) at once. **QED**

Remark 5 The result is also true for $a=1$. Indeed, in [Ri] it was proved that

$$\mathbb{E}\left[\exp-\sqrt{2}t^{3/2}\int_0^1|b_u|du\right]=\sqrt{\pi t}\sum_{n=1}^{\infty}\frac{\exp(a'_nt)}{-a'_n},\text{ for }t>0.\tag{37}$$

By similar arguments as we used in the end of the preceding proof, we can deduce that (27) is also verified for $a=1$.

Proof of Theorem 2:
i) First let us review some notations and results concerning the Hölder spaces (see also, [B-R]). Take $0<\alpha<1$ and we shall denote $\mathcal{C}^\alpha$ the set of all real paths f on [0,1] which are α-Hölder continuous. The norm of this space will be denoted by

$$\|f\|_\alpha:=\sup_{0\leq t<s\leq 1}\frac{|f(t)-f(s)|}{|t-s|^\alpha}.$$

By $\mathcal{C}_0^\alpha$ we denote the subspace of $\mathcal{C}^\alpha$ of these paths f such that:

$$\lim_{\delta\to 0}\sup_{|t-s|\leq\delta\ ;\ 0\leq t<s\leq 1}\frac{|f(t)-f(s)|}{|t-s|^\alpha}=0.$$

We also denote c_0 the space of all sequences $(\xi_n)_{n\geq 0}$ such that $\lim_{n\to\infty}\xi_n = 0$ endowed with the sup-norm. It is classical (see, for instance, [Ci]) that the separable Banach spaces c_0 and $\mathcal{C}_0^\alpha$ are isomorphic with the isomorphism $T^\alpha : c_0 \longrightarrow \mathcal{C}_0^\alpha$ given by:

$$\xi \longmapsto \sum_{n=1}^{\infty} \frac{\xi_n}{c_n(\alpha)} \Phi_n.$$

Here $\{\Phi_n : n \geq 0\}$ is the Schauder basis and $c_n(\alpha) = 2^{\bar{n}(\alpha-1/2)}2^{1-\alpha}$ with $\bar{n} = \lfloor \log_2 n \rfloor$. Recall that, since the standard Brownian bridge has its paths a.s. in the α-Hölder space ($\alpha < 1/2$) it can be decomposed in the Schauder basis as:

$$b_u = \sum_{n=2}^{\infty} g_n \Phi_n(u), \quad u \in [0,1],$$

where g_n are independent copies of a standard Gaussian random variables.
ii) Assume $x \geq 0$. The expectation in the expression (5) of the density $p_t^\varepsilon(x)$ can be bounded:

$$\mathbb{E}\left[\exp - \left\{ \frac{\gamma t}{2}\int_0^1 |xs - \varepsilon\sqrt{t}b_s|^{-1/2}ds + \frac{t}{2\varepsilon}\int_0^1 |xs - \varepsilon\sqrt{t}b_s|ds \right\}\right] \leq E,$$

where

$$E := \mathbb{E}\left[\exp - \frac{t}{2\varepsilon}\int_0^1 |xs - \varepsilon\sqrt{t}b_s|ds\right].$$

Recall that ϕ_0 is the function which minimises the functional A given by (15), hence

$$p_t^\varepsilon(x) \leq \frac{E}{\varepsilon\sqrt{2\pi t}} \exp \frac{A(\phi_0)}{2\varepsilon^2}.$$

Let us introduce the Hölder ball $B_\alpha(\phi_0, \beta) = \{\phi \in \mathcal{C}_0^\alpha : \|\phi - \phi_0\|_\alpha \leq \beta\}$ ($\beta > 0$) and we split $E = E_1 + E_2$ as follows:

$$E_1 := \mathbb{E}\left[\exp\left\{-\frac{t}{2\varepsilon^2}\int_0^1 |xs - \varepsilon\sqrt{t}b_s|ds\right\} \mathbb{1}_{\{\varepsilon b \notin B_\alpha(\phi_0,\beta)\}}\right]$$

and

$$E_2 := \mathbb{E}\left[\exp\left\{-\frac{t}{2\varepsilon^2}\int_0^1 |xs - \varepsilon\sqrt{t}b_s|ds\right\} \mathbb{1}_{\{\varepsilon b \in B_\alpha(\phi_0,\beta)\}}\right].$$

iii) Let us consider E_1: by the Varadhan principle for the Brownian bridge in Hölder norm, we get

$$\limsup_{\varepsilon\to 0} 2\varepsilon^2 \ln E_1 = - \inf_{\{\phi\in H^1\cap B(\phi_0,\beta)^c:\phi(0)=\phi(1)=0\}} A(\phi) \leq -A(\phi_0) - \beta. \tag{38}$$

Indeed, this can be done using the following:

Lemma 1 *For any continuous function ψ on $[0,1]$ such that $\psi(0) = \psi(1) = 0$,*

$$A(\psi) - A(\phi_0) \geq t^{3/2}\int_0^a |\psi(u) - \phi_0(u)|du + \|\psi - \phi_0\|_{H^1}. \tag{39}$$

Here the functional A is given by (15) *and the point a is defined in* (17). *Moreover,*

$$if \ \|\psi - \phi_0\|_\alpha \geq \beta, \ then \ A(\psi) - A(\phi_0) \geq \beta.$$

We postpone the proof of Lemma 1 and we finish the proof of Theorem 2.
iv) For the function ϕ_0 we can write:

$$\phi_0(u) := \sum_{n=1}^{\infty} \frac{\xi_n}{c_n(\alpha)} \Phi_n(u) \text{ and } \|\phi_0\|_{H^1} = \sum_{n=1}^{\infty} \frac{\xi_n^2}{c_n^2(\alpha)}. \tag{40}$$

Let us consider E_2:

$$E_2 = \lim_{N\uparrow\infty} \mathbb{E}\left[\exp - \left\{ \frac{t}{2\varepsilon^2} \int_0^1 \left| xs - \varepsilon\sqrt{t} \sum_{n=2}^{N} g_n \Phi_n(s) \right| ds \right\} \prod_{n=2}^{N} \mathbb{I}_{\{\varepsilon c_n(\alpha) g_n \in [\xi_n - \beta, \xi_n + \beta]\}} \right]$$

$$= \lim_{N\uparrow\infty} \int \dots \int_{\prod_{n=2}^{N} [\frac{\xi_n-\beta}{c_n(\alpha)}, \frac{\xi_n+\beta}{c_n(\alpha)}]} \frac{\varepsilon^{-N}}{(2\pi)^{N/2}} \exp - \left\{ \frac{t}{2\varepsilon^2} \int_0^1 \left| xs - \varepsilon\sqrt{t} \sum_{n=2}^{N} x_n \Phi_n(s) \right| ds \right.$$

$$\left. + \sum_{n=2}^{N} \frac{x_n^2}{2\varepsilon^2} \right\} dx_1 \dots dx_N.$$

By Lemma 1 and by (40), we get

$$E_2 \leq \lim_{N\uparrow\infty} \left(\exp - \frac{A(\phi_0)}{2\varepsilon^2} \right) \int \dots \int_{\prod_{n=2}^{N} [\frac{\xi_n-\beta}{c_n(\alpha)}, \frac{\xi_n+\beta}{c_n(\alpha)}]} \frac{\varepsilon^{-N}}{(2\pi)^{N/2}} \exp - \left\{ \sum_{n=2}^{N} \frac{(c_n(\alpha)x_n - \xi_n)^2}{2\varepsilon^2 c_n(\alpha)^2} \right.$$

$$\left. - \frac{t^{3/2}}{2\varepsilon^2} \int_0^a \left| \sum_{n=2}^{N} (c_n(\alpha)x_n - \xi_n) \frac{\Phi_n(s)}{c_n(\alpha)} \right| ds \right\} dx_1 \dots dx_N,$$

since $\phi_0(u) = xu/\sqrt{t}$ on [0,a] (see also **i)** in the proof of Proposition 5). Then, again by the decomposition of the Brownian bridge, we deduce

$$E_2 \leq \left(\exp - \frac{A(\phi_0)}{2\varepsilon^2} \right) \mathbb{E}\left[\exp - \frac{t^{3/2}}{2\varepsilon} \int_0^a |b_s| ds \right]. \tag{41}$$

Therefore, by (38) and by Proposition 7,

$$\limsup_{\varepsilon\to 0} \varepsilon^{2/3} \ln p_t^{\varepsilon}(x) \leq \max\left\{ -\infty \ , \ a_1' \left(\frac{t}{2} - \sqrt{|x|} \right) \right\} = a_1' \left(\frac{t}{2} - \sqrt{|x|} \right),$$

This ends the proof of the theorem except for the proof of Lemma 1. **QED.**

Proof of Lemma 1:
i) We assume without loss of generality, that $x \geq 0$ and let us denote by $\tilde{\psi}$ the following function:

$$\tilde{\psi}(u) := \psi(u) - 2\left(\psi(u) - \frac{xu}{\sqrt{t}} \right) \mathbb{I}_{\{\psi(\bullet) - \frac{x\bullet}{\sqrt{t}} \geq 0\}}(u).$$

Then

$$t\int_0^1 |xu - \sqrt{t}\psi(u)| du = t\int_0^1 |xu - \sqrt{t}\tilde{\psi}(u)| du,$$

hence

$$A(\psi) = t\int_0^1 |xu - \sqrt{t}\tilde{\psi}(u)| du + \int_0^1 \psi'^2(u) du.$$

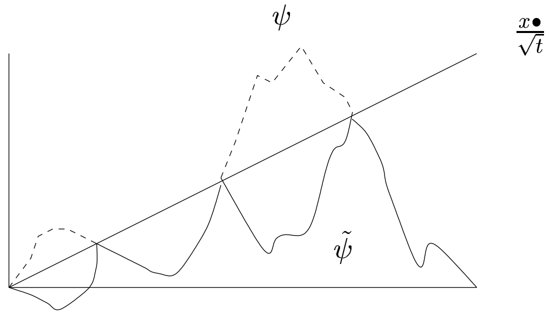

Figure 4: Definition of $\tilde{\psi}$

Since $x\bullet - \sqrt{t}\tilde{\psi}(\bullet) \geq 0$, we obtain

$$\begin{aligned} A(\psi) &= t\int_0^1 (xu - \sqrt{t}\tilde{\psi}(u))du + \int_0^1 \psi'^2(u)du \\ &= t\int_0^1 (xu - \sqrt{t}\phi_0(u))du + \int_0^1 (\phi_0')^2(u)du - t^{3/2}\int_0^1 (\tilde{\psi} - \phi_0)(u)du \\ &\quad +2\int_0^1 (\phi_0'(\psi' - \phi_0'))(u)du + \int_0^1 (\psi' - \phi_0')^2(u)du \\ &= A(\phi_0) - t^{3/2}\int_0^1 (\tilde{\psi} - \phi_0)(u)du - 2\int_0^1 \phi_0''(\psi - \phi_0)(u)du + \int_0^1 (\psi' - \phi_0')^2(u)du \\ &= A(\phi_0) - t^{3/2}\int_0^a (\tilde{\psi} - \phi_0)(u)du + \int_0^1 (\psi' - \phi_0')^2(u)du - t^{3/2}\int_a^1 (\tilde{\psi} - \psi)(u)du. \end{aligned}$$

Here we used the fact that $\phi_0''(u) = -t^{3/2}/2$ on $[a,1]$ and $\phi_0''(u) = 0$ on $[0,a]$. Then we get (39) by a simple lower bound.
ii) Clearly, for $f \in \mathcal{C}_0^\alpha$, $\|f\|_{H^1} \geq \|f\|_\alpha$. Then, for ψ in the complementary of the Hölder ball $B_\alpha(\phi_0,\beta)$, we can write $A(\psi) - A(\phi_0) \geq \beta$. **QED**

5. VISCOSITY SOLUTION OF A HAMILTON-JACOBI EQUATION

In Theorem 1 we obtained the behaviour of $p_t^\varepsilon(x)$, if (t,x) does not lie between the extremals. The aim of this section is to study the behaviour for (t,x) lying between the extremals, namely we study $s(\varepsilon)\ln p_t^\varepsilon(x)$, with

$$s(x) = |x|^{(2(1-\gamma))/(1+\gamma)}.$$

Let us note that if $\gamma = 1/2$, $s(\varepsilon) = \varepsilon^{2/3}$ and we already obtained some information in Theorem 2. The improvement of this last result is contained in the following:

Theorem 3 *If (t,x) belongs to the domain contained between the extremal solutions of* (3), *then*

$$\lim_{\varepsilon\to 0} s(\varepsilon)\ln p_t^\varepsilon(x) = -\lambda_1\left(t - \frac{|x|^{1-\gamma}}{1-\gamma}\right). \tag{42}$$

Here λ_1 is the first positive eigenvalue of the Schrödinger operator on $L^2(\mathrm{I\!R})$:

$$-\frac{1}{2}\frac{d^2}{dx^2} + \frac{\gamma}{2|x|^{1-\gamma}} + \frac{|x|^{2\gamma}}{2}.$$

Our study is based on a particular tool: the viscosity solutions of parabolic partial differential equations. For a study of these solutions the reader may consult the book of Barles [B].
First we shall introduce some domains of the first quadrant plane:

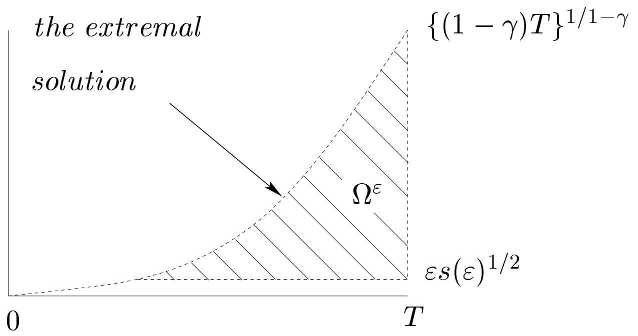

Figure 5: The domain Ω^ε

$$\begin{aligned}
\Omega &:= \{(t,x): \quad 0<t<T \quad , \quad 0<x<\{(1-\gamma)T\}^{1/1-\gamma}\},\\
\tilde{\Omega} &:= \{(t,x): \quad 0<t\leq T \quad , \quad 0<x<\{(1-\gamma)T\}^{1/1-\gamma}\},\\
\widehat{\Omega} &:= \{(t,x): \quad 0<t\leq T \quad , \quad 0\leq x<\{(1-\gamma)T\}^{1/1-\gamma}\},\\
\Omega^\varepsilon &:= \{(t,x): \quad (1-\gamma)\varepsilon^{4/(1+\gamma)}<t<T \quad , \quad \varepsilon s(\varepsilon)^{1/2}<x<\{(1-\gamma)T\}^{1/1-\gamma}\},\\
\tilde{\Omega}^\varepsilon &:= \{(t,x): \quad (1-\gamma)\varepsilon^{4/(1+\gamma)}<t\leq T \quad , \quad \varepsilon s(\varepsilon)^{1/2}<x<\{(1-\gamma)T\}^{1/1-\gamma}\},\\
\widehat{\Omega}^\varepsilon &:= \{(t,x): \quad (1-\gamma)\varepsilon^{4/(1+\gamma)}<t\leq T \quad , \quad \varepsilon s(\varepsilon)^{1/2}\leq x<\{(1-\gamma)T\}^{1/1-\gamma}\}.
\end{aligned}$$

Let us consider the following parabolic partial differential equation:

$$\frac{\partial u}{\partial t}+H(t,x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2})=0 \quad \text{in } U\subset \mathbb{R}^2, \tag{43}$$

where H is a real Hamiltonian defined on $\mathbb{R}\times U\times\mathbb{R}\times\mathbb{R}$. We assume that H is elliptic in the following sense:

$$H(t,x,u,p,q_1)\leq H(t,x,u,p,q_2), \text{ if } q_2\leq q_1.$$

We recall the notion of viscosity solution for (43) and we need a slightly different definition than the one in [B] (see Definition 2.1, p. 11 or Definition 4.1, p. 80), since the domains which we consider do not be open or closed sets.

Definition 1 *Let u be a bounded upper semi-continuous (u.s.c.) (respectively lower semi-continuous (l.s.c.)) function on a connected set U with connected boundary. u is a viscosity sub-solution (respectively super-solution) of (43) on U, if for all $\varphi\in\mathcal{C}^2(U)$, whenever $(t_0,x_0)\in U$ is a point of local maximum (local minimum) of $u-\varphi$, then*

$$\frac{\partial\varphi}{\partial t}(t_0,x_0)+H(t_0,x_0,u(t_0,x_0),\frac{\partial\varphi}{\partial x}(t_0,x_0),\frac{\partial^2\varphi}{\partial x^2}(t_0,x_0))\leq 0 \ \ (\textit{respectively} \ \ \geq 0). \tag{44}$$

Proposition 8 *Let us define*

$$u^\varepsilon(t,x):=-s(\varepsilon)\ln(p_t^\varepsilon(x)+e^{-D/s(\varepsilon)}), \tag{45}$$

where $D>0$. Then u^ε is a viscosity solution of

$$\frac{\partial u^\varepsilon}{\partial t}+H_\varepsilon(t,x,u^\varepsilon,\frac{\partial u^\varepsilon}{\partial x},\frac{\partial^2 u^\varepsilon}{\partial x^2})=0 \qquad \textit{in } \tilde{\Omega}^\varepsilon, \tag{46}$$

corresponding to the Hamiltonian:

$$H_\varepsilon(t,x,u,p,q) := -\frac{\varepsilon^{\frac{4\gamma}{1+\gamma}}}{2}q + \frac{\varepsilon^{\frac{4\gamma}{1+\gamma}}}{2}p^2 + x^\gamma p - \gamma x^{\gamma-1}s(\varepsilon)\left(1-\exp\frac{u-D}{s(\varepsilon)}\right). \tag{47}$$

Remark 6 The reason to introduce the exponential term, with $D>0$, in the definition of u^ε is that this last function is bounded. Clearly, by choosing D large enough, this term in the logarithm scale will not change the limit as $\varepsilon \to 0$.

Proof of Proposition 8:

a) First, we shall prove that the equation (46) is verified on Ω^ε.
If V is regular, the function of (t,x)

$$(t,x) \mapsto \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t})\mathbb{E}\left[\exp -\frac{1}{2}\int_0^t V(B_s)ds\middle| B_t = x\right]$$

is a classical solution of the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}V\,u \quad \text{on }]0,T]\times \mathbb{R}^*.$$

This is also true if we assume Hölder continuity conditions on V (see [Ro] Theorem 1, p.127). Since V is Lipschitz in the domain $\mathbb{R}_+^*$ and $\mathbb{R}_-^*$, then $p_\cdot^\varepsilon(\cdot) \in \mathcal{C}^{1,2}(\Omega^\varepsilon)$. Therefore, we deduce by a logarithmic transformation, that u^ε is a classical solution of

$$\frac{\partial u}{\partial t} + H_\varepsilon(t,x,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2}) = 0 \quad \text{on } \Omega^\varepsilon,$$

where H_ε is given by (47).

b) Let us prove now that u^ε given by (45) is a viscosity solution. Take $\varphi \in \mathcal{C}^2(\Omega^\varepsilon)$ and let $(t_0,x_0)\in\Omega^\varepsilon$ be a local maximum of $u^\varepsilon - \varphi$. Then

$$\frac{\partial u^\varepsilon}{\partial t}(t_0,x_0) = \frac{\partial \varphi}{\partial t}(t_0,x_0) \quad , \quad \frac{\partial u^\varepsilon}{\partial x}(t_0,x_0) = \frac{\partial \varphi}{\partial x}(t_0,x_0) \quad ,$$

$$\frac{\partial^2 u^\varepsilon}{\partial x^2}(t_0,x_0) - \frac{\partial^2 \varphi}{\partial x^2}(t_0,x_0) \le 0.$$

We deduce

$$\frac{\partial \varphi}{\partial t}(t_0,x_0) + H_\varepsilon(t_0,x_0,u^\varepsilon(t_0,x_0),\frac{\partial \varphi}{\partial x}(t_0,x_0),\frac{\partial^2 \varphi}{\partial x^2}(t_0,x_0)) \le 0.$$

By the same argument we prove that, if (t_0,x_0) is a local minimum of $u^\varepsilon - \varphi$ then

$$\frac{\partial \varphi}{\partial t}(t_0,x_0) + H_\varepsilon(t_0,x_0,u^\varepsilon(t_0,x_0),\frac{\partial \varphi}{\partial x}(t_0,x_0),\frac{\partial^2 \varphi}{\partial x^2}(t_0,x_0)) \ge 0.$$

In fact, this means that all classical solutions are viscosity solutions.

c) Take now $\varphi \in \mathcal{C}^2(\tilde\Omega^\varepsilon)$ and let $(T,x_0)\in\tilde\Omega^\varepsilon$ be a local maximum of $u^\varepsilon-\varphi$.
Replacing φ by $\varphi + (x-x_0)^4 + (t-T)^2$ the first and the second derivative at (T,x_0) do not change, and so we can assume that (T,x_0) is a point of local strict maximum. The idea is to adapt the reasoning for the points of Ω^ε to the point (T,x_0). To do this, we need the following :

Lemma 2 *Let $(u_\eta)_\eta$ be a sequence of u.s.c. functions which converges towards u, uniformly over all compact subsets of U. We suppose that u can be extended to an u.s.c. function on $\bar U$. If (t_0,x_0) is a local strict maximum of u then there exists $(t_\eta,x_\eta)\in\bar U$ such that* $\lim_{\eta\to 0}(t_\eta,x_\eta) = (t_0,x_0)$.

We can finish the proof of Proposition 8. Let us consider the function:

$$\Xi_\eta(t,x) := u^\varepsilon(t,x) - \varphi(t,x) - \frac{\eta}{T-t}, \quad \eta > 0.$$

By Lemma 2, there exists a local maximum $(t_\eta, x_\eta) \in \tilde{\Omega}^\varepsilon$ of Ξ_η which converges to (T, x_0), as $\eta \to 0$. Clearly, $\lim_{t\to T} \Xi_\eta(t,x) = -\infty$. Hence $t_\eta < T$ and for η small enough $(t_\eta, x_\eta) \in \Omega^\varepsilon$. Since u^ε is a viscosity sub-solution on Ω^ε (see **b)**) we get:

$$\frac{\eta}{(T-t)^2} + \frac{\partial \varphi}{\partial t}(t_\eta, x_\eta) + H_\varepsilon(t_\eta, x_\eta, u^\varepsilon(t_\eta, x_\eta), \frac{\partial \varphi}{\partial x}(t_\eta, x_\eta), \frac{\partial^2 \varphi}{\partial x^2}(t_\eta, x_\eta)) \leq 0.$$

By the continuity of u^ε, letting $\eta \to 0$ we obtain

$$\frac{\partial \varphi}{\partial t}(T, x_0) + H_\varepsilon(T, x_0, u^\varepsilon(T, x_0), \frac{\partial \varphi}{\partial x}(T, x_0), \frac{\partial^2 \varphi}{\partial x^2}(T, x_0)) \leq 0.$$

The same argument can be used to prove that u^ε is a super-solution. This ends the proof of Proposition 8 except for the proof of Lemma 2. **QED**

Proof of Lemma 2: The result is clear for $(t_0, x_0) \in U$. Let us suppose that $(t_0, x_0) \in \partial U$. Take $r > 0$ and we define the compact set $K^r = \overline{B((t_0, x_0), r)} \cup \overline{U}$, where B is an Euclidean ball centred in (t_0, x_0) with radius r. The u.s.c. function u_η reaches its maximum on the compact set K^r at (t_η, x_η). We extract a sub-sequence, denoted for simplicity again by (t_η, x_η), which converges to $(\bar{t}, \bar{x})$, as $\eta \to 0$.
Assume that $(\bar{t}, \bar{x}) \notin \partial U$. Since u is u.s.c. and since (t_0, x_0) is a strict maximum, there exists $(t, y) \in U$ such that

$$u(t_0, x_0) > u(t, y) > u(\bar{t}, \bar{x})$$

This inequality can not be true ! Indeed, $u_\eta(t_\eta, x_\eta)$ tends to $u(\bar{t}, \bar{x})$ and $u_\eta(y, t)$ tends to $u(y, t)$, these two convergences being uniform.
Hence, $(\bar{t}, \bar{x}) \in \partial U$. Moreover, we know that $\|(\bar{t}, \bar{x}) - (t_0, x_0)\| \leq r$. We can choose a sequence $(\bar{t}^r, \bar{x}^r)$ which tends to (t_0, x_0), as $r \to 0$. By diagonalization, we can find a sequence $(t_\eta, x_\eta) \in \bar{U}$ which converges to (t_0, x_0), as $\eta \to 0$. **QED**

Our aim is to take the limit as $\varepsilon \to 0$ in the Hamilton-Jacobi equation (46). We prove the following stability result:

Proposition 9 *Let us denote*

$$\bar{u}(t,x) := \limsup_{\varepsilon \to 0,\ s\to t,\ y \to x,\ (s,y) \in \widehat{\Omega}^\varepsilon} u^\varepsilon(s,y), \quad \textit{for all } (t,x) \in \widehat{\Omega}. \tag{48}$$

Then $\bar{u}$ is a viscosity sub-solution of the equation

$$\frac{\partial u}{\partial t} + H_0\left(x, \frac{\partial u}{\partial x}\right) = 0 \quad \textit{on } \tilde{\Omega}, \tag{49}$$

with the Hamiltonian

$$H_0(x,p) := x^\gamma p. \tag{50}$$

If we denote $\underline{u} = \liminf u^\varepsilon$, with a limit taken as previously, then $\underline{u}$ is a viscosity super-solution of (49).

To prove this result we need the following:

Lemma 3 *Let* $(v_\varepsilon)_\varepsilon$ *be a sequence of u.s.c. functions having a local uniform bound on* $\widehat{\Omega}^\varepsilon$. *Let us denote by* $\bar{v} = \limsup v_\varepsilon$ *as in* (48). *We assume that* $\bar{v}$ *has a local strict maximum on* $\widehat{\Omega}$. *Then, there exists a sub-sequence* $(v_{\varepsilon'})_{\varepsilon'}$ *of* $(v_\varepsilon)_\varepsilon$ *and a sequence* $(r_{\varepsilon'}, z_{\varepsilon'})_{\varepsilon'} \in \widehat{\Omega}^\varepsilon$ *such that: for all* $\varepsilon' > 0$, $v_{\varepsilon'}$ *reaches a local maximum on* $\widehat{\Omega}^\varepsilon$ *at* $(r_{\varepsilon'}, z_{\varepsilon'})$ *and*

$$\lim_{\varepsilon'\to 0}(r_{\varepsilon'}, z_{\varepsilon'}) = (r, z),\ \lim_{\varepsilon'\to 0} v_{\varepsilon'}(r_{\varepsilon'}, z_{\varepsilon'}) = \bar{v}(r, z).$$

The proof of this lemma is similar to the proof of Lemma 4.2, p. 88 in [B].

Proof of Proposition 9: Take $\varphi \in \mathcal{C}^2(\tilde{\Omega})$ and let $(t,x) \in \tilde{\Omega}$ be a local maximum of $u^\varepsilon - \varphi$. We apply Lemma 3 to the sequence $v_\varepsilon = u^\varepsilon - \varphi$ and to $\bar{v} = \bar{u} - \varphi$. Hence, there exists a sequence $(u^{\varepsilon'})_{\varepsilon'}$ and a sequence $(t_{\varepsilon'}, x_{\varepsilon'})$, such that for all $\varepsilon' > 0$, $u^{\varepsilon'} - \varphi$ reaches a local maximum on $\tilde{\Omega}^{\varepsilon'}$ at $(t_{\varepsilon'}, x_{\varepsilon'})$ and $u^{\varepsilon'}(t_{\varepsilon'}, x_{\varepsilon'})$ tends to $\bar{u}(t,x)$. Since $(t,x) \in \tilde{\Omega}$, we see that, for $\varepsilon' > 0$ small enough, $(t,x) \in \tilde{\Omega}^{\varepsilon'}$.
By the fact that $u^{\varepsilon'}$ is a viscosity sub-solution we get:

$$\frac{\partial \varphi}{\partial t}(t_{\varepsilon'}, x_{\varepsilon'}) + H^{\varepsilon'}(t_{\varepsilon'}, x_{\varepsilon'}, u^{\varepsilon'}(t_{\varepsilon'}, x_{\varepsilon'}), \frac{\partial \varphi}{\partial x}(t_{\varepsilon'}, x_{\varepsilon'}), \frac{\partial^2 \varphi}{\partial x^2}(t_{\varepsilon'}, x_{\varepsilon'})) \leq 0.$$

Clearly by (47) and (50), for $x > 0$,

$$\lim_{\varepsilon'\to 0} H_{\varepsilon'}(t, x, u, p, q) = H_0(x, p)$$

and taking the limit as $\varepsilon' \to 0$ we obtain:

$$\frac{\partial \varphi}{\partial t}(t,x) + H_0(x, \frac{\partial \varphi}{\partial x}(t,x)) \leq 0.$$

QED

We also prove an uniqueness result contained in the following:

Proposition 10 *For all* $(t,x) \in \widehat{\Omega}$,

$$\bar{u}(t,x) = \underline{u}(t,x). \tag{51}$$

Proof of Proposition 10:
i) First, we prove (51) for $(t,x) \in]0,T] \times \{0\}$. Take $\varphi \in \mathcal{C}^2(\widehat{\Omega})$ and let $(t_0, 0)$ be a local maximum of $\bar{u} - \varphi$. By Lemma 3, there exists a sequence of points $(t_\varepsilon, x_\varepsilon)$ of local maximum of u^ε on $\widehat{\Omega^\varepsilon}$ such that:

$$\lim_{\varepsilon\to 0}(t_\varepsilon, x_\varepsilon) = (t_0, 0).$$

We take a sub-sequence if necessary and we study then two different situations:
a) either $(t_\varepsilon, x_\varepsilon) \in \tilde{\Omega}^\varepsilon$ and taking the limit as $\varepsilon \to 0$ in equation (46) we get

$$\frac{\partial \varphi}{\partial t} - 1 \leq 0; \tag{52}$$

b) or $(t_\varepsilon, x_\varepsilon) \in \widehat{\Omega^\varepsilon} \setminus \tilde{\Omega}^\varepsilon$. Since, for D large enough,

$$u^\varepsilon(t_\varepsilon, x_\varepsilon) = -s(\varepsilon)\ln(p^\varepsilon_{t_\varepsilon}(\varepsilon s(\varepsilon)^{1/2}) + e^{-D/s(\varepsilon)})$$

tends to $\lambda_1 t_0$ as $\varepsilon \to 0$ (see Corollary 2), we get $\bar{u}(t_0, 0) = \lambda_1 t_0$.
We shall prove that the first situation is impossible. Let us consider the following function:

$$\bar{u}(t,x) - \varphi_0(t,x) := \bar{u}(t,x) - \frac{x}{\eta} - \eta \cosh(\frac{t-t_0}{\eta}) - \frac{t-t_0}{\eta}, \tag{53}$$

and let us denote by (t_η, x_η) the point of maximum of $\bar{u} - \varphi_0$ on $\widehat{\Omega}$. Hence

$$(\bar{u} - \varphi_0)(t_\eta, x_\eta) \geq (\bar{u} - \varphi_0)(0, t_0) = \bar{u}(0, t_0) - \eta.$$

Since $\bar{u}$ is bounded, we obtain, by (53) $\lim_{\eta \to 0} x_\eta = 0$ and $\lim_{\eta \to 0} t_\eta = t_0$.
Since $\bar{u}$ is a viscosity sub-solution, if $(t_\eta, x_\eta) \in \tilde{\Omega}$, we get

$$\frac{\partial \varphi_0}{\partial t}(t_\eta, x_\eta) + x_\eta^\gamma \frac{\partial \varphi_0}{\partial x}(t_\eta, x_\eta) \leq 0. \tag{54}$$

Clearly, by (53),

$$\frac{\partial \varphi_0}{\partial t}(t_\eta, x_\eta) = \sinh(\frac{t_\eta - t_0}{\eta}) + \frac{1}{\eta} \text{ and } \frac{\partial \varphi_0}{\partial x}(t_\eta, x_\eta) = \frac{1}{\eta}.$$

It is obvious that equation (52) can not be verified for φ_0. Hence $(t_\eta, x_\eta) \notin \tilde{\Omega}$ and so, for η small enough $(t_\eta, x_\eta) \in]0, T] \times \{0\}$. Moreover, $\bar{u}(t_\eta, 0) = \lambda_1 t_\eta$.
We deduce that

$$\bar{u}(t_0, 0) \leq \lambda_1 t_\eta + \eta - \varphi_0(t_\eta, 0).$$

As $\eta \to 0$ we get

$$\bar{u}(t_0, 0) \leq \lambda_1 t_0.$$

Using the same reasoning for $\underline{u}$ we obtain that $\bar{u} = \underline{u}$ on $]0, T] \times \{0\}$.
ii) Second, we prove (51) for $(t, x) \in \tilde{\Omega}$. Take $\delta > 0$ and we consider the following compact set:

$$K_\delta = \{(t,x) : 0 \leq x \leq ((t-\delta)(1-\gamma))^{1/1-\gamma}\} \cap \tilde{\Omega}.$$

We shall prove that

$$\frac{\partial \varphi}{\partial t}(t,x) + x^\gamma \frac{\partial \varphi}{\partial x}(t,x) \leq 0 \quad \text{ on the boundary } \{(t,x) : x = \{(t-\delta)(1-\gamma)\}^{1/1-\gamma}\} \cap \tilde{\Omega}. \tag{55}$$

Take $\varphi \in \mathcal{C}^2(K_\delta)$ and let $(t, \{(t-\delta)(1-\gamma)\}^{1/1-\gamma})$ be a local maximum of $\bar{u} - \varphi$ on K_δ. Let us consider the sequence of functions

$$\Xi_\eta(t,x) := \bar{u}(t,x) - \varphi(t,x) + \frac{\eta}{x^{1-\gamma} - (t-\delta)(1-\gamma)}.$$

By Lemma 1, there exists a sequence (t_η, x_η) which tends to $(t, \{(t-\delta)(1-\gamma)\}^{1/1-\gamma})$. Let us note that Ξ_η reaches a local maximum on K_δ at (t_η, x_η). Indeed, from the fact that $\Xi_\eta(t,x) \to -\infty$ when (t,x) come closer to the boundary $x = \{(t-\delta)(1-\gamma)\}^{1/1-\gamma}$, we deduce that, for η small enough $(t_\eta, x_\eta) \in \mathrm{int} K_\delta$. Since $\bar{u}$ is a viscosity sub-solution,

$$\frac{\partial \varphi}{\partial t}(t_\eta, x_\eta) - \frac{(1-\gamma)\eta}{(x_\eta^{1-\gamma} - (t_\eta - \delta)(1-\gamma))^2} + x_\eta^\gamma \left(\frac{\partial \varphi}{\partial x}(t_\eta, x_\eta) + \frac{(1-\gamma) x_\eta^{-\gamma}}{(x_\eta^{1-\gamma} - (t_\eta - \delta)(1-\gamma))^2} \right) \leq 0,$$

hence

$$\frac{\partial \varphi}{\partial t}(t_\eta, x_\eta) + x_\eta^\gamma \frac{\partial \varphi}{\partial x}(t_\eta, x_\eta) \leq 0.$$

Letting $\eta \to 0$ we obtain (55).
iii) Finally, let us prove (51) on K_δ. We shall compute

$$M := \sup_{K_\delta} (\bar{u} - \underline{u}). \tag{56}$$

Let us assume that $M > 0$ and, as we have already seen, this maximum can not be reached for $x = 0$.
Take $\alpha > 0$. The function

$$\bar{u}_\alpha(t,x) := \bar{u}(t,x) - \alpha t$$

is a "strict" sub-solution of the equation

$$\frac{\partial u}{\partial t} + H_0(x, \frac{\partial u}{\partial x}) = 0,$$

or, more precisely, it is a sub-solution of the equation

$$\frac{\partial u}{\partial t} + H_0(x, \frac{\partial u}{\partial x}) + \alpha = 0.$$

Let us denote

$$\Psi_\eta(t,s,x,y) := \bar{u}_\alpha(t,x) - \underline{u}(s,y) - \frac{(x-y)^2}{\eta^2} - \frac{(t-s)^2}{\eta^2}, \tag{57}$$

and let $(t_\eta, s_\eta, x_\eta, y_\eta)$ be a point where Ψ_η reaches a local maximum. Then $\bar{u}_\alpha - \chi_1$ reaches a local maximum at (t_η, x_η), where χ_1 denotes the function

$$\chi_1(t,x) := \underline{u}(s_\eta, y_\eta) + \frac{(x-y_\eta)^2}{\eta^2} + \frac{(t-s_\eta)^2}{\eta^2}.$$

By the same argument, $\chi_2 - \underline{u}$ reaches a local maximum at $(s_{\eta'}, y_{\eta'})$ where χ_2 denotes the function

$$\chi_2(s,y) := \bar{u}_\alpha(t_\eta, x_\eta) - \frac{(x_\eta - y)^2}{\eta^2} - \frac{(t_\eta - s)^2}{\eta^2}.$$

To finish the proof we need the following:

Lemma 4 *Let us denote $M_\eta := \sup_{K_\delta} \Psi_\eta$, with Ψ_η given by (57). There exists $\rho > 0$ and a sequence of points $(t_\eta, s_\eta, x_\eta, y_\eta)$ of maximum of Ψ_η, such that*

$$\lim_{\eta \to 0} M_\eta = 0, \tag{58}$$

$$\lim_{\eta \to 0} (x_\eta - y_\eta)^2/\eta^2 = 0, \tag{59}$$

$$x_\eta > \rho \quad \textit{and} \quad y_\eta > \rho \quad \textit{for} \quad \eta \quad \textit{small enough.} \tag{60}$$

We return to the proof of Proposition 10. By Lemma 4, there exists a sub-sequence $(t_{\eta'}, s_{\eta'}, x_{\eta'}, y_{\eta'})$ of $(t_\eta, s_\eta, x_\eta, y_\eta)$, such that $x_{\eta'} > 0$. Since $\bar{u}_\alpha$ is a viscosity sub-solution, we get

$$\frac{\partial \chi_1}{\partial t}(t_{\eta'}, x_{\eta'}) + H_0\left(x_{\eta'}, \frac{\partial \chi_1}{\partial x}(t_{\eta'}, x_{\eta'})\right) + \alpha \leq 0,$$

hence

$$\frac{2(t_{\eta'} - s_{\eta'})}{\eta'^2} + H_0\left(x_{\eta'}, \frac{2(x_{\eta'} - y_{\eta'})}{\eta'^2}\right) + \alpha \le 0. \tag{61}$$

Since $\underline{u}$ is a viscosity super-solution we get

$$\frac{\partial \chi_2}{\partial s}(s_{\eta'}, y_{\eta'}) + H_0\left(y_{\eta'}, \frac{\partial \chi_2}{\partial y}(s_{\eta'}, y_{\eta'})\right) \ge 0,$$

hence

$$\frac{2(t_{\eta'} - s_{\eta'})}{\eta'^2} + H_0(y_{\eta'}, \frac{2(x_{\eta'} - y_{\eta'})}{\eta'^2}) \ge 0. \tag{62}$$

Subtracting (62) from (61) we obtain

$$-\theta_{\eta'}(x_{\eta'}^{\gamma} - y_{\eta'}^{\gamma}) = H_0(x_{\eta'}, \theta_{\eta'}) - H_0(y_{\eta'}, \theta_{\eta'}) \le -\alpha,$$

where

$$\theta_{\eta'} := \frac{2(x_{\eta'} - y_{\eta'})}{\eta'^2}.$$

Taking the limit as $\eta \to 0$, and using (59) and (60), we get

$$0 \le -\alpha.$$

This is in contradiction with the assumption $\alpha > 0$. The proof is complete except for the proof of Lemma 4. **QED**

Remark 7. Obviously,

$$u(t,x) = \lambda_1\left(t - \frac{x^{1-\gamma}}{1-\gamma}\right)$$

is a classical solution of (49), which verify $u(t,0) = \lambda_1 t$. Hence, by the proof of Proposition 10, we deduce an uniqueness result and we get that $\bar{u} = \underline{u} = u$.

Proof of Lemma 4: Put $M_\eta = \sup_{K_\delta} \Psi_\eta = \Psi_\eta(t_\eta, s_\eta, x_\eta, y_\eta)$. Then, by (57), for any (t,x) and (s,y) belonging to K_δ,

$$\bar{u}_\alpha(t,x) - \underline{u}(s,y) - \frac{(x-y)^2}{\eta^2} - \frac{(t-s)^2}{\eta^2} \le M_\eta.$$

Taking $(t,x) = (s,y)$ we get

$$(\bar{u}_\alpha - \underline{u})(t,x) \le M_\eta,$$

hence

$$M = \sup_{K_\delta}(\bar{u} - \underline{u}) \le M_\eta.$$

Since

$$M \le \bar{u}_\alpha(t_\eta, x_\eta) - \underline{u}(s_\eta, y_\eta) - \frac{(x_\eta - y_\eta)^2}{\eta^2} - \frac{(t_\eta - s_\eta)^2}{\eta^2},$$

and by the fact that M, $\bar{u}_\alpha$ and $\underline{u}$ are bounded, there exists $k > 0$, such that

$$\frac{(x_\eta - y_\eta)^2}{\eta^2} + \frac{(t_\eta - s_\eta)^2}{\eta^2} \le k, \quad \text{for all } \eta > 0.$$

We can extract a sub-sequence $(t_{\eta'}, s_{\eta'}, x_{\eta'}, y_{\eta'})$ which converges to $(t,s,x,y) \in K_\delta$ and such that $\left\{(x_{\eta'} - y_{\eta'})^2/\eta'^2 : \eta' \ge 0\right\}$ converges. Since $x_{\eta'} - y_{\eta'} \to 0$ and $t_{\eta'} - s_{\eta'} \to 0$, as $\eta' \to 0$,

we deduce that $t = s$ and $x = y$.
Furthermore,

$$M \le \liminf M_{\eta'} \le \limsup M_{\eta'}$$

$$\le \bar{u}_\alpha(t,x) - \underline{u}(t,x) - \lim_{\eta' \to 0} \frac{(x_{\eta'} - y_{\eta'})^2}{\eta'^{\,2}} - \liminf_{\eta' \to 0} \frac{(t_{\eta'} - s_{\eta'})^2}{\eta'^{\,2}} \le M.$$

Hence

$$\lim_{\eta' \to 0} M_{\eta'} = M \quad \text{ and } \lim_{\eta' \to 0} \frac{(x_{\eta'} - y_{\eta'})^2}{\eta'^{\,2}} = 0.$$

(60) is clear, since $\bar{u} = \underline{u}$ on $]0,T] \times \{0\}$. **QED**

Finally, by the symmetry of $p_t^\varepsilon(\cdot)$, the result of Theorem 3 is an easy consequence of Remark 6 and of the following:

Proposition 11 *For* $(t,x) \in \tilde{\Omega}$*,*

$$\lim_{\varepsilon \to 0} s(\varepsilon) \ln p_t^\varepsilon(x) = \lambda_1 \left(\frac{x^{1-\gamma}}{1-\gamma} - t \right),$$

and the convergence is uniform on each compact subset of $\tilde{\Omega}$*. Moreover,*

$$\lim_{\varepsilon \to 0} s(\varepsilon) \ln p_t^\varepsilon(0) = -\lambda_1 t, \quad \forall t > 0.$$

Proof of the Proposition 11: This proof is an adaptation of the proof of a result in [B] (see Lemma 4.1, p. 86).
Let K be a compact subset of $\tilde{\Omega}$. First, we show that $\lim_{\varepsilon \to 0} u^\varepsilon = u$, uniformly on K. By Proposition 10, $\bar{u} = \underline{u} = u$ on K. This means that u is a continuous function (since $\bar{u}$ is u.s.c. and $\underline{u}$ is l.s.c.). Hence, by (45) $u^\varepsilon - u$ is also a continuous function and $M_\varepsilon := \sup_K (u^\varepsilon - u)$ is reached at $(t_\varepsilon, x_\varepsilon) \in K$.
Since u^ε is bounded, we can extract a sub-sequence $(t_{\varepsilon'}, x_{\varepsilon'})$, such that $(t_{\varepsilon'}, x_{\varepsilon'}) \to (t,x) \in K$ and $M_{\varepsilon'} \to (\limsup_\varepsilon M_\varepsilon)$, as $\varepsilon' \to 0$.
By (48) we get

$$\limsup_{\varepsilon' \to 0} u^{\varepsilon'}(t_{\varepsilon'}, x_{\varepsilon'}) \le \bar{u}(t,x).$$

Hence

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \quad \sup_K (u^\varepsilon - u) \quad &= \quad \limsup_{\varepsilon' \to 0} \left(u^{\varepsilon'}(t_{\varepsilon'}, x_{\varepsilon'}) - u(x_{\varepsilon'}, t_{\varepsilon'}) \right) \\
&\le \quad \bar{u}(t,x) - u(t,x) = 0.
\end{aligned}$$

By similar arguments, we obtain:

$$\limsup_{\varepsilon \to 0} \quad \sup_K (u - u^\varepsilon) \le 0.$$

Therefore,

$$\limsup_{\varepsilon \to 0} \quad \sup_K (u - u^\varepsilon) = 0.$$

On the other hand,

$$\limsup_{\varepsilon \to 0} u^\varepsilon(t,x) = \inf \left(\limsup_{\varepsilon \to 0} (-s(\varepsilon) \ln p_t^\varepsilon(x)), D \right)$$

$$\liminf_{\varepsilon\to 0} u^\varepsilon(t,x) = \inf\left(\liminf_{\varepsilon\to 0}(-s(\varepsilon)\ln p_t^\varepsilon(x)), D\right).$$

We deduce that, for D large enough, the term $\exp(-D/s(\varepsilon))$ will not change the limits as ε tends to zero, since $u(t,x) = \lambda_1(t - \frac{x^{1-\gamma}}{1-\gamma})$ is bounded. Hence, $s(\varepsilon)\ln p_t^\varepsilon(x)$ converges uniformly on each compact set of $\tilde{\Omega}$ to $\lambda_1(\frac{x^{1-\gamma}}{1-\gamma} - t)$.
Finally, for $x = 0$, we use formula (10) which we proved in Proposition 5:

$$p_t^\varepsilon(0) = \frac{1}{\varepsilon s(\varepsilon)^{1/2}} \sum_{j=1}^{\infty} e^{-\frac{\lambda_j t}{s(\varepsilon)}} \psi_j^2(0).$$

By a similar reasoning as in the proof of Corollary 2, we can show that:

$$\lim_{\varepsilon\to 0} s(\varepsilon)\ln p_t^\varepsilon(0) = -\lambda_1 t.$$

QED

ACKNOWLEDGEMENTS

The authors are grateful to Saïd Benachour for its suggestions of references for §5. This paper originate in a stimulating discussion with Youssef Ouknine on the Ouarzazate road.

REFERENCES

[A-S] M. Abramowitz and A. Stegun, *Pocket book of mathematical functions*, Verlag Harri Deutsch, 1984.

[B-B] R. Bafico and P. Baldi, *Small Random Perturbations of Peano Phenomena*, Stochastics, **6**, (1982), pp. 279-292.

[B-R] P. Baldi and B. Roynette, *Some exact equivalents for the Brownian motion in Holder norm*, Probab. Theory Relat. Fields, **93** (1992), pp. 457-484.

[B] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, Springer Verlag, 1994.

[B-R-V] S. Benachour, B. Roynette and P. Vallois, *Solutions fondamentales de* $u_t - \frac{1}{2}u_{xx} = \pm|u_x|$, Hommage à P. A. Meyer et J. Neveu, pp. 41-71, Astérisque, 1996.

[Ca] R. Carmona, *Regularity Properties of Schrödinger and Dirichlet Semigroups*, J. Funct. Analysis, **33** (1979), pp. 259-296.

[Ci] Z. Ciesielski, *On the Isomorphisms of the Spaces* H_α *and* m, Bull. Acad. Pol. Sci., **8** (1960), pp. 217-222.

[D1] E. B. Davies, *Properties of the Green's functions of some Schrödinger operators*, J. London Math. Soc.(2), **7** (1973), pp. 483-491.

[D2] E. B. Davies, *One-parameter semigroups*, Academic Press, 1980

[D-S] E. B. Davies and B. Simon, *Ultracontractivity and the Heat Kernel for Schrödinger Operators and Dirichlet Laplacians*, J. Funct. Analysis, **59** (1984), pp. 335-395.

[De-S] J. D. Deuschel and D. W. Stroock, *Large Deviations*, Academic Press, 1989

[F] A. Friedman, *Partial differential equations of parabolic type*, Printice-Hall, 1964.

[I-W] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd edition, North-Holland Kodansha, 1989.

[I-MK] K. Ito and H. P. McKean, *Diffusion Processes and their Sample Paths*, 2nd printing, Springer Verlag, 1974.

[K] M. KAC, *On some connections between probability theory and differential and integral equations*, Proceedings of the Second Berkeley Symposium. of Math. Statist. Probab. 1950, pp. 189-215, University of California Press, 1951.

[K-S] I. KARATZAS and S. E. SHREVE, *Brownian Motion and Stochastic Calculus*, 2nd edition, Springer Verlag, 1991.

[R-S] M. REED and B. SIMON, *Methods of Modern Mathematical Physics III: Scattering Theory*, Academic Press, 1979.

[R-Y] D. REVUZ and M. YOR, *Continuous Martingales and Brownian motion*, 2nd edition, Springer Verlag, 1994

[Ri] S. O. RICE, *The integral of the absolute value of the pinned Wiener process calculation of its probability density by numerical integration*, Ann. Prob., **10** (1982), pp. 240-243.

[Ro] M. ROSENBLATT, *On a class of Markov Processes*, Trans. Amer. Math. Soc., **71** (1951), pp. 120-135.

[Sh] L. A. SHEPP, *On the integral of the absolute value of the pinned Wiener process*, Ann. Prob., **10** (1982), pp. 234-239.