Rate of convergence of some self-attracting diffusions

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Abstract: We consider the rate of convergence of the paths of some self-attracting diffusions. We prove that these diffusions are attracted by their *mean*-process. Moreover, under some assumptions on the interaction function f, this attraction becomes strong enough to imply the almost sure convergence of the paths. In this case, we provide an estimate of the rate of convergence which depends on the growth of f near the origin.

Keywords: self-attracting diffusion, comparison result, long memory process, stopping time

M. Cranston and Y. Le Jan introduced self-attracting diffusions in [1]. These processes, attracted by their own paths, satisfy a stochastic differential equation of the form

$$X_t = B_t + \int_0^t \int_0^s f(X_s - X_u) du\, ds, \tag{1}$$

where B is a one dimensional Brownian motion and f a decreasing function. For f measurable and locally bounded, (1) admits a unique weak solution [2]. Since these processes are self-attracting, it is particularly interesting to describe the asymptotic behaviour of their paths. Some earlier works developed results on the boundedness and convergence of such paths. M. Cranston and Y. Le Jan [1] studied two particular cases: linear and constant interaction, S. Herrmann and B. Roynette generalized these results [2]. In fact, if f is strictly decreasing in a neighbourhood of the origin and satisfies some weak conditions there, the paths converge almost surely. The aim of this paper is to present some rate of convergence of the paths. The main result is contained in the following

Theorem 1 *If f is an odd decreasing function such that $f \in \mathcal{C}^1(\mathbb{R})$ and if there exist $\eta > 0$, $\gamma \geq 1$ and $C_\gamma > 0$ such that*

$$|f(x) - f(y)| \geq C_\gamma |x-y|^\gamma, \quad \textit{for all} \ \ |x-y| \leq \eta, \tag{2}$$

then, for any $\mu < 1/1+\gamma$,

$$\lim_{t\to\infty} \left\{ \left(\frac{t}{\ln t}\right)^\mu \sup_{s\geq t} |X_s - X_t| \right\} = 0 \quad a.s. \tag{3}$$

The proof of Theorem 1 is based on comparison results. In the first section, we present some preliminary results on convergence and rates of convergence for some Markov diffusions having a non homogeneous drift term. Then, in the second section, we prove convergence of the self-attracting diffusion towards the *mean*-process. Finally we conclude the proof of Theorem 1 and give a simple proof of the rate of convergence in a particular case: the linear one.

1 Preliminary results

The method used to prove the almost sure convergence of the paths are based on local comparisons between the solution of the self-attracting process and a suitable Markov process (see, for instance, [1] and [2]). It turns out that this main key is also crucial to obtain an estimate of the rate of convergence. So, in this section, we shall describe the asymptotic behaviour of a particular Markov process which will play a leading role in next section.
Let Φ be a non-decreasing function. We consider the following SDE:

$$X_t = B_t - \int_0^t s\Phi(X_s)ds, \tag{4}$$

where B is a one dimensional Brownian motion. Since the drift of this diffusion is locally bounded, (4) admits a unique weak solution. Moreover, the diffusion coefficient is constant, so that $L^0(X^{(1)} - X^{(2)}) = 0$. Here L_0 is the local time at the origin and $(X^{(1)}, X^{(2)})$ is any pair of solutions. Hence we deduce that (4) admits a unique strong solution (see, for instance, [6] Proposition 3.2 p.389). We are especially interested in asymptotic properties of X as t becomes large. These properties depend on the behaviour of Φ near the origin, in the following way.

Proposition 1 *a) If* Φ *is non-decreasing,* $\Phi(x) > 0$ *if* $x > 0$, $\Phi(x) < 0$ *if* $x < 0$ *then* $\lim_{t\to\infty} X_t = 0 \quad a.s.$
b) Moreover, if there exist $\eta > 0$, $\gamma > 0$ *and* $C_\gamma > 0$ *such that*

$$|\Phi(x)| \geq C_\gamma |x|^\gamma, \quad \textit{for} \quad |x| \leq \eta, \tag{5}$$

then there exists $K_\gamma > 0$ *such that*

$$\limsup_{t\to\infty} \left\{ \left(\frac{t}{\ln t}\right)^{1/1+\gamma} \sup_{s\geq t} |X_s| \right\} \leq K_\gamma \quad a.s. \tag{6}$$

Remark: One can choose $K_\gamma = 2\left(\frac{3}{C_\gamma}\right)^{1/1+\gamma}$.
Proof: Let us consider the sequence of stopping times defined by:

$$\begin{aligned} &\tau_0 = 0, \\ &\tau_{n+1} = \inf\{t \geq \tau_n + n : \; X_t = 0\}. \end{aligned}$$

This proof is divided into two parts. In the first part, we shall get some estimate for the rate of convergence of X on the intervals $[\tau_n, \tau_n + 2n]$. In the second part, we shall establish that, for large n, $\tau_{n+1} \leq \tau_n + 2n$, so that the estimate holds for all t large enough.
i) Let us compare the solution squared of (4) to a diffusion whose drift is independent of time. Let $Y^{(n)}$ be the solution of the equation:

$$Y_t^{(n)} = 2\int_{\tau_n}^t \sqrt{Y_s^{(n)}} dW_s - 2\alpha_n \int_{\tau_n}^t \sqrt{Y_s^{(n)}} \widetilde{\Phi}_n\left(\sqrt{Y_s^{(n)}}\right) ds + t - \tau_n, \quad t \geq \tau_n, \tag{7}$$

where $\alpha_n = \sum_{k=0}^{n-1} k = n(n-1)/2$, W is a Brownian motion defined by

$$dW_t = \mathrm{sgn}(X_t) dB_t$$

and $\widetilde{\Phi}_n$ is some non-decreasing odd Lipschitz function satisfying

$$\widetilde{\Phi}_n(x) = \begin{cases} 0 & \text{if} \quad |x| < \varepsilon_n, \\ h_n & \text{if} \quad x \geq (1+\delta)\varepsilon_n, \\ -h_n & \text{if} \quad x \leq -(1+\delta)\varepsilon_n, \end{cases}$$

for some arbitrary $\delta > 0$. Here $h_n := \Phi(\varepsilon_n) \wedge (-\Phi(-\varepsilon_n))$ and $(\varepsilon_n)_{n\geq 0}$ is a sequence of positive reals which decreases to zero. $Y^{(n)}$ is well defined since the drift is Lipschitz. Indeed, there exists a unique strong solution to (7) (see, for instance [4] Proposition 5.2.13 p.291)

Let us note that, for $t \geq \tau_n$ and $x \in \mathbb{R}$,

$$-\alpha_n x \widetilde{\Phi}_n(x) \geq -t x \Phi(x). \tag{8}$$

Hence we obtain the following comparison result:

Lemma 1
$$\mathbb{P}(X_t^2 \le Y_t^{(n)}, \quad \forall t \ge \tau_n) = 1$$

We postpone the proof of this lemma.
Using Lemma 1, we are able to compare the probabilities of some particular events. We define the events:

$$E_n = \left\{ \sup_{[\tau_n, \tau_n + 2n]} |X_t| \ge (2+\delta)\varepsilon_n \right\} \tag{9}$$

We get

$$\mathbb{P}(E_n) \le \mathbb{P}\left(\sup_{[\tau_n,\, \tau_n + 2n]} \sqrt{Y_t^{(n)}} \ge (2+\delta)\varepsilon_n \right).$$

Let us note that $(Y_{t+\tau_n}^{(n)})_{t\ge 0}$ has the same law as $\{(U_t^{(n)})^2,\ t \ge 0\}$, the unique week solution of the following S.D.E:

$$U_t^{(n)} = B_t - \alpha_n \int_0^t \widetilde{\Phi}_n(U_s^{(n)}) ds.$$

Hence, by the symmetry of $\widetilde{\Phi}_n$,

$$\begin{aligned} \mathbb{P}\left(\sup_{[\tau_n,\, \tau_n + 2n]} \sqrt{Y_t^{(n)}} \ge (2+\delta)\varepsilon_n \right) &= \mathbb{P}\left(\sup_{[0,\, 2n]} |U_t^{(n)}| \ge (2+\delta)\varepsilon_n \right) \\ &= 2\mathbb{P}\left(\sup_{[0,2n]} U_t^{(n)} \ge (2+\delta)\varepsilon_n \right). \end{aligned}$$

Furthermore, by a classical comparison result (see Theorem 1.1 chapter 6 in [3]), we get

$$\mathbb{P}(E_n) \le 2\mathbb{P}\left(\sup_{[0,\, 2n]} Z_t^{(n)} \ge \varepsilon_n \right).$$

Here

$$Z_t^{(n)} = B_t - \alpha_n h_n \int_0^t \mathrm{sgn}(Z_s^{(n)}) ds.$$

Let us define the stopping time $T_n = \inf\{t \ge 0,\ Z_t^{(n)} = \varepsilon_n\}$. Hence, using Markov's inequality, we get

$$\mathbb{P}(E_n) \le 2e^2 \mathbb{E}[e^{-\frac{T_n}{n}}].$$

In order to compute the Laplace transform $\mathbb{E}[e^{-\lambda_n T_n}]$, it suffices to solve the following differential equation (see, for instance, [1] or [5])

$$\frac{1}{2} f''_{\lambda_n}(x) - \alpha_n h_n f'_{\lambda_n}(x) = \lambda_n f_{\lambda_n}(x).$$

We obtain

$$\mathbb{E}[e^{-\lambda_n T_n}] = \left\{\cosh(\varepsilon_n\sqrt{\beta_n^2+2\lambda_n}) + \left(1-\frac{2\beta_n}{\sqrt{\beta_n^2+2\lambda_n}}\right)\sinh(\varepsilon_n\sqrt{\beta_n^2+2\lambda_n})\right\}^{-1} e^{-\beta_n\varepsilon_n}, \quad (10)$$

with $\beta_n = \alpha_n h_n$. So, if $n\beta_n^2$ tends to infinity as n becomes large, there exists a constant $K > 0$ such that

$$\mathbb{P}(E_n) \leq K n \alpha_n^2 h_n^2 e^{-2\alpha_n h_n \varepsilon_n}, \quad n \geq 1. \quad (11)$$

We may thus choose some sequence $(\varepsilon_n)_{n\geq 0}$ (for instance, such that $\varepsilon_n \to 0$ and $\varepsilon_n h_n \geq 1/n$) in such a way that:

$$\sum_{n=0}^{\infty} \mathbb{P}(E_n) < \infty$$

and, by Borel-Cantelli's lemma we obtain that, a.s., ω belongs to a finite number of E_n only.
Furthermore, if Φ satisfies Condition (5), then it suffices to take $\varepsilon_n = \left(\frac{(6+\delta)\ln n}{C_\gamma n^2}\right)^{1/1+\gamma}$, for some arbitrary $\delta > 0$, to get the convergence of the series.
ii) The second step of the proof consists in establishing: a.s., $\omega \in \Omega$ belongs only to a finite number of A_n defined by

$$A_n = \{w \in \Omega \text{ s.t. } X_t \neq 0, \ \forall t \in [\tau_n + n, \tau_n + 2n]\}.$$

Using a comparison result as in **i)** and a scaling property, we obtain

$$\mathbb{P}(A_n) \leq \mathbb{P}(V_t^{(n)} \neq 0 \text{ on } [n\alpha_n^2, 2n\alpha_n^2]),$$

where

$$V_t^{(n)} = B_t - \int_0^t \psi_n(V_s^{(n)})ds$$

with ψ_n some non-decreasing Lipschitz function satisfying

$$\psi_n(x) = \begin{cases} 0 \text{ if } |x| < \alpha_n, \\ \Phi(1) \text{ if } x \geq 2\alpha_n, \\ -\Phi(-1) \text{ if } x \leq -2\alpha_n. \end{cases}$$

Using the Markov property, we get

$$\mathbb{P}(A_n) \leq \int_{\mathbb{R}} \mathbb{P}_x^{V^{(n)}}(T_0 \geq n\alpha_n^2) f_{n\alpha_n^2}(x)dx,$$

where T_0 is the first hitting time by the process $V^{(n)}$ of the level 0 and $f_t(x)$ is the density of $V_t^{(n)}$. Let us choose a sequence of numbers $\xi_n > 0$ such that, as n tends to infinity, we get $\alpha_n/\xi_n \to 0$. Then we obtain

$$\mathbb{P}(A_n) \leq \mathbb{P}(|V_{n\alpha_n^2}^{(n)}| \geq \xi_n) + \mathbb{P}_{\xi_n}^{V^{(n)}}(T_0 \geq n\alpha_n^2).$$

Setting

$$R_t = B_t - \Phi(1)\int_0^t \operatorname{sgn}(R_s)ds,$$

we obtain

$$\begin{aligned} \mathbb{P}\left(|V_{n\alpha_n^2}^{(n)}| \geq \xi_n\right) &\leq 2\mathbb{P}\left(\sup_{[0,\,n\alpha_n^2]} V_t^{(n)} \geq \xi_n\right) \leq 2\mathbb{P}\left(\sup_{[0,\,n\alpha_n^2]} R_t \geq \xi_n - 2\alpha_n\right) \\ &\leq 2\mathbb{P}(T_{\xi_n - 2\alpha_n} \leq n\alpha_n^2) \leq 2e\,\mathbb{E}\left[\exp\left\{-\frac{1}{n\alpha_n^2}T_{\xi_n-2\alpha_n}\right\}\right]. \end{aligned}$$

Here we denote by T_a the first hitting time of level a by the diffusion R. By (10) and (11), we obtain, for n large enough, the existence of a constant $K > 0$ such that

$$\mathbb{P}\left(|V_{n\alpha_n^2}^{(n)}| \geq \xi_n\right) \leq K\exp\left\{-\Phi(1)\xi_n\right\}.$$

Hence we obtain:

$$\sum_{n\geq 0}\mathbb{P}\left(|V_{n\alpha_n^2}^{(n)}| \geq \xi_n\right) < \infty.$$

On the other hand,

$$\mathbb{P}_{\xi_n}^{V^{(n)}}(T_0 \geq n\alpha_n^2) \leq \mathbb{P}_{\xi_n-2\alpha_n}^{R}(T_0 \geq n\alpha_n^2).$$

Using the method developed in [1] and [5] to compute the Laplace transform of T_0 (hitting time of level 0), we obtain the following density for T_0 as the process starts from x :

$$p_x(t) = \frac{|x|}{\sqrt{2\pi t^3}}\, e^{-\frac{t\Phi(1)^2}{2} - \frac{x^2}{2t} + |x\Phi(1)|}. \tag{12}$$

So we easily deduce that (for instance by choosing $\xi_n = \alpha_n\sqrt{n}$)

$$\sum_{n\geq 0}\mathbb{P}_{\xi_n}^{V^{(n)}}(T_0 \geq n\alpha_n^2) < \infty.$$

Hence, by Borel-Cantelli's lemma we get that a.s. $\omega \in \Omega$ belongs only to a finite number of A_n. Together with i) this proves part a) of Proposition 1.
iii) In order to prove statement b), let us choose a random variable N such that, for $n \geq N$,

$$\tau_{n+1} < \tau_n + 2n \quad \text{and} \quad \sup_{t \in [\tau_n, \tau_n + 2n]} |X_t| < (2+\delta)\varepsilon_n.$$

Then, for $t \geq \tau_N$, there exists $m \in \mathbb{N}$ such that $\tau_m \leq t < \tau_{m+1}$. Hence

$$\sup_{s \geq t} |X_s| \leq (2+\delta)\varepsilon_m = (2+\delta)\left(\frac{(6+\delta)\ln m}{C_\gamma m^2}\right)^{1/1+\gamma}.$$

We thus obtain for $t \geq e \vee \tau_N$:

$$\left(\frac{t}{\ln t}\right)^{1/1+\gamma} \sup_{s \geq t} |X_s| \leq (2+\delta)\left(\frac{(6+\delta)\ln m}{\ln(2\alpha_{m+1} + \tau_N)} \frac{2\alpha_{m+1} + \tau_N}{C_\gamma m^2}\right)^{1/1+\gamma}.$$

As t tends to infinity, the upper bound in the preceding inequality is equivalent to

$$(2+\delta)\left(\frac{6+\delta}{2C_\gamma}\right)^{1/1+\gamma}.$$

δ being arbitrary, we let it tend to zero and thus obtain the bound announced in the remark. □

Proof of Lemma 1:
This lemma is a classical result if the drift term is Lipschitz (see for instance Theorem 1.1 p.437 in [3]).
Let us consider a decreasing sequence of real numbers $(a_k)_{k \in \mathbb{N}}$ such that $a_0 = 1$, $\lim_{k \to \infty} a_k = 0$ and $\int_{a_k}^{a_{k-1}} \frac{dx}{x} = k$. For all $k \in \mathbb{N}$, there exists a continuous function ρ_k such that $\text{supp}(\rho_k) \subset (a_k, a_{k-1})$, $0 \leq \rho_k(x) \leq \frac{2}{kx}$ and $\int_{a_k}^{a_{k-1}} \rho_k(x)dx = 1$.
Let us then define

$$\psi_k(x) = \int_0^x \int_0^y \rho_k(u)dudy, \quad x \in \mathbb{R}.$$

By Itô's formula, we get

$$\begin{aligned}
\psi_k(X_t^2 - Y_t^{(n)}) &= 2\int_{\tau_n}^t \psi_k'(X_s^2 - Y_s^{(n)})(|X_s| - \sqrt{Y_s^{(n)}})\text{sgn}(X_s)dB_s \\
&-2\int_{\tau_n}^t \psi_k'(X_s^2 - Y_s^{(n)})\left(sX_s\Phi(X_s) - \alpha_n\sqrt{Y_s^{(n)}}\widetilde{\Phi}_n(\sqrt{Y_s^{(n)}})\right) ds \\
&+2\int_{\tau_n}^t \psi_k''(X_s^2 - Y_s^{(n)})\left(|X_s| - \sqrt{Y_s^{(n)}}\right)^2 ds
\end{aligned}$$

Since $\psi_k'(x) > 0 \Longrightarrow x \geq a_k$ and,

$$sX_s\Phi(X_s) \geq \alpha_n\sqrt{Y_s^{(n)}}\widetilde{\Phi}(\sqrt{Y_s^{(n)}}) \quad \text{for} \quad s \geq \tau_n \, (\geq \alpha_n)$$

if $|X_s| \geq \sqrt{Y_s^{(n)}}$, we get

$$\begin{aligned} & \mathbb{E}[\psi_k(X_t^2 - Y_t^{(n)})\mathbb{1}_{\{\tau_n \leq t\}}] \\ \leq \; & 2\mathbb{E}\int_{\tau_n}^{t\vee\tau_n} \psi_k''(X_s^2 - Y_s^{(n)})(|X_s| - \sqrt{Y_s^{(n)}})^2 ds \\ \leq \; & \frac{4}{k}\mathbb{E}\int_{\tau_n}^{t\vee\tau_n} \frac{(|X_s| - \sqrt{Y_s^{(n)}})^2}{(X_s^2 - Y_s^{(n)})} ds \leq \frac{4(t\vee\tau_n)}{k}. \end{aligned}$$

Letting k tend to infinity we obtain $\mathbb{E}[(X_t^2 - Y_t^{(n)})_+] = 0$, which implies the comparison result. □

2 Convergence to the *mean*-process

The aim of this section is to prove that, under weak conditions, self-attracting diffusions stay near the *mean*-process. Let us define this random process. The self-attracting diffusion is defined by (1) where f is a strictly decreasing function, $f \in \mathcal{C}^1(\mathbb{R})$ such that $f(0) = 0$. In this section we do not need to assume that f is odd. We denote by μ_t the occupation measure of the diffusion at time t. The drift term of equation (1) can be expressed as

$$D(t,x) := \int_{\mathbb{R}} f(x-y)d\mu_t(y). \tag{13}$$

Since f is decreasing and $f \in \mathcal{C}^1(\mathbb{R})$, the drift term is in $\mathcal{C}^1(\mathbb{R})$ for the variable x and strictly decreasing almost surely, in case $t > 0$.

Definition 1 *The* mean-*process C_t is the process satisfying the equation $D(t, C_t) = 0$ for $t > 0$ and $C_0 = X_0$.*

Proposition 2 *a) If there exist $c > 0$ and $R > 0$ such that*

$$|f'(x)| \geq c \quad when \quad |x| \geq R, \tag{14}$$

then $\lim_{t\to\infty} |X_t - C_t| = 0$ a.s.
b) If f satisfies (2) then there exists a constant $K_\gamma > 0$ such that

$$\limsup_{t\to\infty} \left\{ \left(\frac{t}{\ln t}\right)^{1/1+\gamma} \sup_{s\geq t} |X_s - C_s| \right\} \leq K_\gamma \quad a.s. \tag{15}$$

Remark: In the statement of a) we do not give any conditions on the behaviour of f in a neighbourhood of the origin. This means that, even if we do not know whether the process converges or not (see [2]), we obtain a convergence of the difference between the self-attracting diffusion and the *mean*-process. When the speed of convergence is fast enough, we can also prove an a.s. convergence of the paths.

Proof: Let us denote by $Z_t := X_t - C_t$ the difference between the self-attracting process and the *mean*-process. Using Definition 1 and the regularity of the drift term of equation (1), we get

$$0 = \frac{d}{dt}D(t, C_t) = \frac{\partial D}{\partial x}(t, C_t)C'_t + f(-Z_t).$$

f is decreasing, so that $\frac{\partial D}{\partial x}(t, C_t) \neq 0$ almost surely for $t > 0$. Hence

$$C'_t = \frac{-f(-Z_t)}{\frac{\partial D}{\partial x}(t, C_t)} = \frac{-f(-Z_t)}{\int_{\mathbb{R}} f'(C_t - y)d\mu_t(y)}, \tag{16}$$

which is of the same sign as Z. We thus obtain that Z satisfies the following equation:

$$\begin{cases} dZ_t = \left(\int_0^t f(X_t - X_s)ds + \frac{f(-Z_t)}{\int_{\mathbb{R}} f'(C_t - y)d\mu_t(y)}\right)dt + dB_t \\ Z_0 = 0 \end{cases} \tag{17}$$

For $\varepsilon > 0$, let us define some function Φ satisfying $\Phi(\varepsilon) \leq \inf_{x\in\mathbb{R}}\{f(x) - f(x+\varepsilon)\}$, $\Phi(-\varepsilon) := -\Phi(\varepsilon)$ and such that $x \to \sqrt{x}\Phi(\sqrt{x})$ is a Lipschitz function. Since f is strictly decreasing and satisfies either the condition (14) or the condition (2), we can choose Φ such that $\Phi(\varepsilon) > 0$ for $\varepsilon > 0$. Moreover the first part of the drift term in (17) becomes

$$\int_0^t f(X_t - X_s)ds = \int_{\mathbb{R}} f(Z_t + C_t - y)d\mu_t(y).$$

Hence

$$\operatorname{sgn}(Z_t)\int_0^t f(X_t - X_s)ds \leq -t\operatorname{sgn}(Z_t)\Phi(Z_t).$$

Since C'_t has the same sign as Z_t, the drift term $\delta(t,x)$ of equation (17) satisfies $\operatorname{sgn}(x)\delta(t,x) \leq -t\operatorname{sgn}(x)\Phi(x)$. This leads to the following comparison result

$$\mathbb{P}(Z_t^2 \leq U_t, \forall t \geq 0) = 1,$$

where the process $\{U_t,\ t \geq 0\}$ has the same law as $\{Y_t^2,\ t \geq 0\}$ with Y_t the solution of the SDE

$$\begin{cases} dY_t = dW_t - t\,\Phi(Y_t)dt, \\ Y_0 = 0. \end{cases} \tag{18}$$

Here W is some Brownian motion. The proof of this result is similar to the proof of Lemma 1. Let us just note that equation (18) admits a unique weak solution, that Y_t^2 satisfies a SDE which admits a unique strong solution due to the regularity of the function $\sqrt{x}\Phi(\sqrt{x})$ and that equation (1) admits a unique strong solution due to the regularity of f (see [2]). To finish the proof we just use Proposition 1: if f satisfies condition (14), then Φ is an odd function, $\Phi(x) > 0$ for $x > 0$ and the first part of the proposition leads to a); if f satisfies condition (2) then the second part of Proposition 1 implies the convergence result (15). □

3 Rate of convergence

In the preceding section, we proved that the difference between the self-attracting diffusion and the *mean*-process tends to zero. If this convergence is fast enough, then we obtain an almost sure convergence of the paths and some estimate on the rate of convergence. First we shall prove Theorem 1 and then we will study a particular case: the linear interaction.

Proof of Theorem 1:
In fact we shall only prove the convergence of some normalized upper bound of $\sup_{s\geq t}(X_s - X_t)$. In order to get the announced pathwise convergence, one then applies similar arguments to some lower bound.
Let us define the sequence of stopping times

$$\begin{cases} \tau_0 = 0 \\ \tau_{n+1} = \inf\{t \geq \tau_n + n : \quad Z_t = 0\} \end{cases}$$

We recall that Z is the difference between the self-attracting diffusion and its *mean*-process. Let $\delta > 0$. Using the proofs of Proposition 1 and 2, we obtain the existence of a random variable N such that, for $n \geq N$, $\tau_{n+1} \leq \tau_n + 2n$ and, for $t \geq \tau_N$,

$$\sup_{s\geq t} |Z_s| \leq (K_\gamma + \delta)\left(\frac{\ln t}{t}\right)^{1/1+\gamma}. \tag{19}$$

For each $m \geq N$, we construct a sequence of stopping times as follows:

$$S_0 \quad = \quad \inf\{s \geq \tau_m, \quad C_s = C_{\tau_m} + \alpha_0^{(m)}\}$$

$$S_k \;=\; \inf\{s \geq S_{k-1},\;\; C_s = C_{S_{k-1}} + \alpha_{k-1}^{(m)} + \alpha_k^{(m)}\} \;\text{ for }\; k \geq 1, \quad (20)$$

with

$$\alpha_0^{(m)} \;=\; (K_\gamma + \delta)\left(\frac{\ln \tau_m}{\tau_m}\right)^{1/1+\gamma}$$

$$\alpha_k^{(m)} \;=\; (K_\gamma + \delta)\left(\frac{\ln S_{k-1}}{S_{k-1}}\right)^{1/1+\gamma}, \quad k \geq 1. \quad (21)$$

We recall that C was defined in Definition 1. To prove the statement of Theorem 1, it suffices to prove that $\lim_{k\to\infty} S_k = \infty$ and that, for $r < 1/1+\gamma$, there exists a constant $K > 0$ such that, for m large enough,

$$\sum_{k \geq 0} \alpha_k^{(m)} \leq K \left(\frac{\ln \tau_m}{\tau_m}\right)^r. \quad (22)$$

By Definition 1, $D(S_k, C_{S_k}) = 0$. Hence

$$0 = \int_{\mathbb{R}} f(C_{S_k} - y) d\mu_{S_k}(y) = I_1 + I_2 + I_3, \quad (23)$$

where $I_1 = \int_{\mathbb{R}} f(C_{S_k} - y) d\mu_{\tau_m}(y)$, $I_2 = \int_{\mathbb{R}} f(C_{S_k} - y)(d\mu_{S_{k-1}} - d\mu_{\tau_m})(y)$ and $I_3 = \int_{\mathbb{R}} f(C_{S_k} - y)(d\mu_{S_k} - d\mu_{S_{k-1}})(y)$. Let us find some upper bound for each of these integrals.

- Since f is decreasing and $C_{S_k} \geq C_{\tau_m}$, we get

$$I_1 \leq \int_{\mathbb{R}} f(C_{\tau_m} - y) d\mu_{\tau_m}(y) = D(\tau_m, C_{\tau_m}) = 0.$$

- Let $S_{-1} = \tau_m$. For $0 \leq i \leq k$ and $s \in [S_{i-1}, S_i]$, the inequality (19) implies that $|Z_s| \leq \alpha_i^{(m)}$. Hence

$$X_s = C_s + Z_s \leq C_s + \alpha_i^{(m)} \leq C_{S_i} + \alpha_i^{(m)}. \quad (24)$$

  By the definition (20) of the hitting times, (24) implies that $X_s \leq C_{S_k} - \alpha_k^{(m)}$ for $s \in [\tau_m, S_{k-1}]$. Since f is decreasing, $f(C_{S_k} - X_s) \leq f(\alpha_k^{(m)})$ a.s. on $[\tau_m, S_{k-1}]$ and

$$I_2 \leq f(\alpha_k^{(m)})(S_{k-1} - \tau_m).$$

- Finally, by (24), on the time interval $[S_{k-1}, S_k]$, we get $f(C_{S_k} - X_s) \leq f(-\alpha_k^{(m)})$ a.s. and thus

$$I_3 \leq f(-\alpha_k^{(m)})(S_k - S_{k-1}).$$

Since f is an odd function, (23) and the combination of the upper bounds of I_1, I_2 and I_3 lead to

$$0 \leq f(\alpha_k^{(m)})(S_{k-1} - \tau_m - (S_k - S_{k-1})). \tag{25}$$

For $k \geq 1$, we get

$$S_k - S_{k-1} \geq S_{k-1} - \tau_m \geq 2^{k-1}(S_0 - \tau_m), \tag{26}$$

which implies $\lim_{k\to\infty} S_k = \infty$. Furthermore, by (26) and for m large enough,

$$\begin{aligned} \sum_{k\geq 0} \alpha_k^{(m)} &\leq (K_\gamma + \delta) \sum_{k\geq 1} \left(\frac{\ln(\tau_m + 2^{k-1}(S_0 - \tau_m))}{\tau_m + 2^{k-1}(S_0 - \tau_m)} \right)^{1/1+\gamma} \\ &+ (K_\gamma + \delta) \left(\frac{\ln \tau_m}{\tau_m} \right)^{1/1+\gamma}. \end{aligned} \tag{27}$$

Setting $a = \tau_m$ and $b = S_0 - \tau_m$, we first prove that b is bounded away from 0 as m becomes large. Using (16), and since $f \in \mathcal{C}^1(\mathbb{R})$, we obtain the existence of some random constants $K_i > 0$ such that, for $t \geq \tau_N$,

$$C'_t \leq K_1 f\left((K_\gamma + \delta) \left(\frac{\ln t}{t} \right)^{1/1+\gamma} \right) \leq K_2 \left(\frac{\ln t}{t} \right)^{1/1+\gamma}.$$

Hence, by definition of b,

$$\alpha_0^{(m)} = \int_{\tau_m}^{S_0} C'_t dt \leq K_2 \left(\frac{\ln \tau_m}{\tau_m} \right)^{1/1+\gamma} b.$$

We obtain the lower bound for b using the definition of $\alpha_0^{(m)}$. Now we are able to estimate the expression (27). Let us note that the function $g(x) = \ln\left(\frac{\ln x}{x}\right)$ is a convex function for $x \geq e^2$, i.e. for all $\nu \in]0, 1[$,

$$g(\nu x + (1-\nu)y) \leq \nu g(x) + (1-\nu) g(y), \quad x, y \geq e^2.$$

So, by computing $\exp g(x)$ and using the fact that $x/\ln x$ is an increasing function on $[e,\infty)$, we deduce that, for $x,y \geq e^2$,

$$\frac{\ln(x+y)}{x+y} \leq \frac{\ln(\nu x+(1-\nu)y)}{\nu x+(1-\nu)y} \leq \left(\frac{\ln x}{x}\right)^{\nu}\left(\frac{\ln y}{y}\right)^{1-\nu}.$$

Set $\nu := r(1+\gamma) < 1$. Let us choose $k_0 \in \mathbb{N}$ such that $2^{k_0}b \geq e^2$, then, for m large enough, there exists a constant $K > 0$, independent of a and b, such that

$$\begin{aligned}\sum_{k\geq 0}\left(\frac{\ln(a+2^k b)}{a+2^k b}\right)^{1/1+\gamma} &\leq k_0\left(\frac{\ln a}{a}\right)^{1/1+\gamma} + \sum_{k\geq k_0}\left(\frac{\ln a}{a}\right)^{r}\left(\frac{\ln(2^k b)}{2^k b}\right)^{\frac{1-\nu}{1+\gamma}} \\ &\leq K\left(\frac{\ln a}{a}\right)^{r}.\end{aligned}$$

This proves (22). Letting $r := \frac{1}{2(1+\gamma)} + \frac{\mu}{2}$, we then have

$$\begin{aligned}\limsup_{t\to\infty}\left(\frac{t}{\ln t}\right)^{\mu}\sup_{s\geq t}|X_s - X_t| &\leq \limsup_{t\to\infty}\left(\frac{t}{\ln t}\right)^{\mu}\left(\sup_{s\geq t}|Z_s| + \sup_{s\geq t}|C_s - C_t|\right) \\ &\leq \limsup_{m\to\infty}\left(\frac{\tau_m}{\ln \tau_m}\right)^{\mu} 2\sum_{k\geq 0}\alpha_k^{(m)} = 0.\end{aligned}$$

□

Remark: The statement of Theorem 1 holds for odd interaction functions. In fact, we can also provide some rate of convergence in the following more general case: f decreasing, $f(0) = 0$, $f \in \mathcal{C}^1(\mathbb{R})$ and f satisfying (2). Nevertheless we obtain a weaker speed of convergence. The ideas of the proof are the following:
we define another sequence of stopping times

$$S_k = \inf\{s \geq S_{k-1}, C_s = C_{S_{k-1}} + \alpha_{k-1}^{(m)} + f^{-1}(-f(-\alpha_k^{(m)}))\}, \text{ for } k \geq 1,$$

where f^{-1} is the inverse function of f. (25) holds for such stopping times, so that $\lim_{k\to\infty} S_k = \infty$. Moreover, since $f \in \mathcal{C}^1(\mathbb{R})$ and f satisfies (2), we know that for m large enough, there exists a constant $K > 0$ such that

$$f^{-1}(-f(-\alpha_k^{(m)})) \leq K\left(\alpha_k^{(m)}\right)^{1/\gamma}.$$

Using similar arguments as those presented in the proof of Theorem 1, we obtain, for some constant $K > 0$,

$$\sum_{k\geq 0}\alpha_k^{(m)} + \sum_{k\geq 0} f^{-1}(-f(-\alpha_k^{(m)})) \leq K\left(\frac{\ln \tau_m}{\tau_m}\right)^{r}, \text{ for } r < \frac{1}{\gamma(1+\gamma)}.$$

We thus obtain that, for $\mu < \frac{1}{\gamma(1+\gamma)}$,

$$\lim_{t\to\infty} \left\{ \left(\frac{t}{\ln t}\right)^{\mu} \sup_{s\geq t} |X_s - X_t| \right\} = 0, \quad a.s.$$

□

Let us note that, in the case of a linear interaction, i.e. when $f(x) = -ax$ for some $a > 0$, the statement of Theorem 1 can be improved.

Proposition 3

$$|X_t - X_\infty| = \mathcal{O}\left(\sqrt{\frac{\ln t}{t}}\right) \quad a.s.\ as\ \ t \to \infty. \tag{28}$$

Proof: Let us denote $R_t := tX_t - \int_0^t X_s ds$ and the *mean*-process $M_t := \frac{1}{t}\int_0^t X_s ds$, for $t \geq 0$. By the variation of constant method, we obtain

$$R_t = e^{-at^2/2} \int_0^t s e^{as^2/2} dB_s,$$

since $dR_t = t\,dX_t = -at\,R_t dt + t\,dB_t$. Let us prove that

$$\frac{|R_t|}{\sqrt{t \ln t}} = \mathcal{O}(1) \quad \text{as} \quad t \to \infty. \tag{29}$$

Using the time-change property for martingales (see, for instance, [4] Theorem 4.6 p. 174), we obtain the existence of a Brownian motion β such that

$$\int_0^t s e^{as^2/2} dB_s = \beta_{\int_0^t s^2 e^{as^2} ds}.$$

By the law of iterated logarithm (see [4] Theorem 9.23 p.112) we get, for $\varepsilon > 0$ and $t \geq t_0(\omega, \varepsilon)$,

$$\begin{aligned}
\frac{|R_t|}{\sqrt{t\ln t}} &\leq (1+\varepsilon) e^{-at^2/2} \frac{\sqrt{2}\sqrt{\int_0^t s^2 e^{as^2} ds \ln\ln \int_0^t s^2 e^{as^2} ds}}{\sqrt{t \ln t}} \\
&\leq \frac{(1+\varepsilon)}{\sqrt{a}} \sqrt{\frac{\ln\ln \int_0^t s^2 e^{as^2} ds}{\ln t}}
\end{aligned}$$

We deduce (29). Furthermore,

$$M'_t = \frac{1}{t}X_t - \frac{1}{t}M_t = \frac{R_t}{t^2} = \mathcal{O}\left(\frac{\sqrt{\ln t}}{t^{3/2}}\right) \quad \text{as} \quad t \to \infty.$$

Hence $|M_t - M_\infty| = \mathcal{O}\left(\sqrt{\frac{\ln t}{t}}\right)$. Since $X_t = \frac{R_t}{t} + M_t$ we obtain (28). □

Moreover the rate of convergence is optimal in the linear case.

Proposition 4 *If $f(x) = -ax$ with $a > 0$, then $X_t \to X_\infty$ a.s. as $t \to \infty$ and*

$$\limsup_{t\to\infty} \sqrt{\frac{t}{\ln t}}(X_t - X_\infty) = -\liminf_{t\to\infty} \sqrt{\frac{t}{\ln t}}(X_t - X_\infty) = \sqrt{\frac{2}{a}} \quad a.s. \qquad (30)$$

Proof: Cranston and Le Jan [1] proved that $X_t \to X_\infty$. Moreover they gave an explicit description of X:

$$X_t = \int_0^t h(t,s)dB_s \quad \text{and} \quad X_\infty = \int_0^\infty h(s)dB_s,$$

where

$$h(t,s) = 1 - ase^{\frac{as^2}{2}} \int_s^t e^{-\frac{au^2}{2}}du \quad \text{and} \quad h(s) = 1 - ase^{\frac{as^2}{2}} \int_s^\infty e^{-\frac{au^2}{2}}du.$$

Let us first decompose the difference $X_t - X_\infty$ as follows

$$X_t - X_\infty = S_t - V_t.$$

Here $S_t = \int_0^t (h(t,s) - h(s))dB_s$ and $V_t = \int_t^\infty h(s)dB_s$.

i) Let us study the asymptotic behaviour of S_t as $t \to \infty$. We get

$$S_t = \left(\int_t^\infty e^{-\frac{au^2}{2}}du\right)\left(\int_0^t ase^{\frac{as^2}{2}}dB_s\right).$$

There exists a Brownian motion β such that

$$\int_0^t ase^{\frac{as^2}{2}}dB_s = \beta_{\int_0^t (as)^2 e^{as^2}ds}.$$

Using the law of iterated logarithm for Brownian paths, and since

$$\int_t^\infty e^{-\frac{au^2}{2}}du \sim \frac{1}{at}e^{-\frac{at^2}{2}} \quad \text{as} \quad t \to \infty,$$

we obtain

$$\limsup_{t\to\infty} \sqrt{\frac{t}{\ln t}} S_t = \sqrt{\frac{2}{a}}.$$

ii) It remains to study V_t. Let us first invert the time scale. We thus define

$$M_t := \int_{\frac{1}{t}}^{\infty} h(s) dB_s \quad \text{and} \quad M_0 = 0.$$

M_t is a martingale with respect to a suitable time-inverse filtration and

$$< M >_t = \int_{\frac{1}{t}}^{\infty} h^2(s) ds$$

is an increasing function. So, in an enlarged filtration (because of the boundedness of $< M >_\infty$), there exists a Brownian motion β such that $M_t = \beta_{<M>_t}$. In order to describe the behaviour of V_t as $t \to \infty$, we shall compute

$$\limsup_{t\to 0} \frac{1}{\sqrt{t \ln \frac{1}{t}}} M_t.$$

Using again the law of iterated logarithm, we get

$$\limsup_{t\to 0} \frac{1}{\sqrt{t \ln \frac{1}{t}}} M_t = \limsup_{t\to 0} \frac{\sqrt{2 < M >_t \ln\ln < M >_t^{-1}}}{\sqrt{t \ln t^{-1}}}.$$

In fact we get the following upper bound: $< M >_t \leq t^3/3a^2$. Indeed, since

$$-\left(e^{-\frac{au^2}{2}} \left(\frac{1}{au} - \frac{1}{a^2u^3}\right)\right)' = e^{-\frac{au^2}{2}} - \frac{3e^{-\frac{au^2}{2}}}{a^2u^4} < e^{-\frac{au^2}{2}},$$

which yields, for $t \geq s$,

$$h(s) \quad \leq \quad h(t,s) \leq 1 + ase^{\frac{as^2}{2}} \left[e^{-\frac{au^2}{2}} \left(\frac{1}{au} - \frac{1}{a^2u^3}\right)\right]_s^t$$

Taking the limit as $t \to \infty$, we get $h(s) \leq 1/as^2$ and obtain the upper bound for $< M >_t$. Moreover, since $x \ln\ln x^{-1}$ is an increasing function in a right neighbourhood of the origin, we obtain

$$\sqrt{2 < M >_t \ln\ln < M >_t^{-1}} \leq \sqrt{\frac{2t^3}{3a^2} \ln\ln \frac{3a^2}{t^3}}.$$

Hence

$$\limsup_{t\to\infty}\sqrt{\frac{t}{\ln t}}V_t = \limsup_{t\to 0}\frac{1}{\sqrt{t\ln t^{-1}}}M_t = 0.$$

iii) The symmetry of the EDS (1) enables one to obtain the lim inf. □

Acknowledgement
We are much indebted to an anonymous referee for very constructive criticism.

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